Final exam: Some problems for review

1. Let $F = \langle P, Q \rangle$ be a vector field defined on the entire plane. What is the condition of P and Q for F to be conservative? $Q_x = P_y$ (note that the entire plane is simply connected)

2. Let $F = \langle P, Q, R \rangle$ be a vector field defined on the entire space. What is the condition of P, Q, R for F to be conservative? curl $F = 0$ ($\mathbb{R}^3$ is simply connected)

3. Let $F = \langle 3x^2 + 2xy, x^2 + 2y \rangle$. Is F a conservative vector field? If so, what is a potential function of F? yes, because $P_y = Q_x$. $f(x,y) = x^3 + x^2y + y^2$

4. If F is a vector field then div F is also a vector field. True or false?

5. If F is a vector field then curl F is also a vector field. True or false?

6. If f is a scalar function then $\text{div}\,(\nabla f) = 0$. True or false? $f(x,y) = x^2+y^2$

7. If F is a vector field then $\text{div}(\text{curl}\, F) = 0$. True or false?

8. If F is a vector field then $\text{curl}(\text{curl}\, F) = 0$. True or false? $f(x,y,z) = \langle y^2, 0, 0 \rangle$

9. If f is a scalar function then $\text{curl}(\nabla f) = 0$. True or false?

10. The spherical coordinates:

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases}$$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

What is the range of θ and ϕ?

11. Convert the Cartesian coordinates $(x, y, z) = (-2, 2, 2\sqrt{6})$ into spherical coordinates.

12. Convert the spherical coordinates $(\rho, \theta, \phi) = (2, \pi/3, \pi/2)$ into Cartesian coordinates.

13. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transform $x = u^2 + uv$, $y = uv^2$.

14. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ of the transform $x = u^2$, $y = v^2$, $z = w^2$.

15. In the spherical coordinates, $dV =$__________$d\rho d\phi d\theta$.

16. Find the volume of the solid enclosed by $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$.

17. Evaluate the integral $\int_C (x^2 + y^2)dx + (x^2 - y^2)dy$ where C is the triangle with vertices at $(0, 0)$, $(2, 1)$, $(0, 1)$. positively oriented.

Prob 11

$\rho = 4\sqrt{2}, \quad \phi = \frac{\pi}{6}, \quad \theta = \frac{3\pi}{4}$

Prob 12

$x = 1, \quad y = \sqrt{3}, \quad z = 0$

Prob 13

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u+v & u \\ v^2 & 2uv \end{vmatrix} = 4u^2v + uv^2$$

Prob 14

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$$

Prob 15

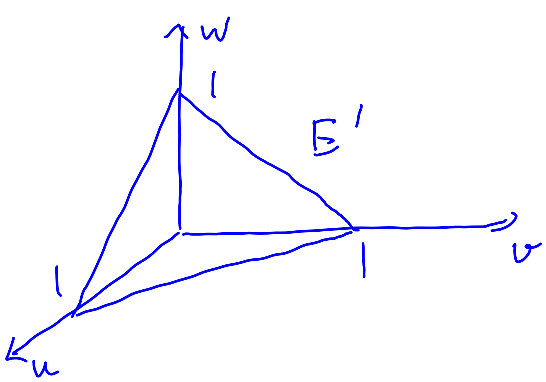
$$dV = \underbrace{\rho^2 \sin\phi}_{\text{Jacobian}} \, d\rho \, d\phi \, d\theta$$

(you can memorize this for the final exam)

Prob 16

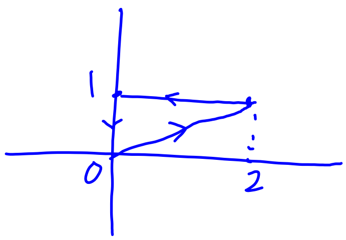
Change of variables: $x = u^2,\ y = v^2,\ z = w^2$

$$E:\ \sqrt{x} + \sqrt{y} + \sqrt{z} \le 1 \ \longrightarrow\ \left.\begin{array}{l} u + v + w \le 1 \\ u, v, w \ge 0 \end{array}\right\} E'$$



$$\text{vol}(E) = \iiint_E 1 \, dV = \iiint_{E'} \underbrace{8uvw}_{\text{Jacobian}} \, dw \, dv \, du = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw \, dw \, dv \, du$$

Prob 17



$$\int_C \underbrace{(x^2+y^2)}_{P}\,dx + \underbrace{(x^2-y^2)}_{Q}\,dy$$

$$\xrightarrow[\text{theorem}]{\text{Green's}} \iint_D (Q_x - P_y)\,dA$$

$$= \iint_D (2x-2y)\,dA$$

$$= \int_0^2 \int_{x/2}^{1} (2x-2y)\,dy\,dx = \cdots$$