Lecture 26

Friday, March 12, 2021 2:29 PM

* Prayer
* Spiritual thought
* Answering questions . . .

Recall:

$$\int_{[a,b]} f(x)\,dx = \int_{[c,d]} f(g(u))\,|g'(u)|\,du$$

$|g'(u)|$ ↑ stretching factor

Now f is a function of two variables: $f = f(x,y)$.

Change of variable $(x,y) \longrightarrow (u,v)$.

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\iint_D f(x,y)\,dx\,dy = \int_{D'} f(x(u,v), y(u,v))\,\boxed{?}\,du\,dv$$

$\boxed{?}$ ↑ stretching factor

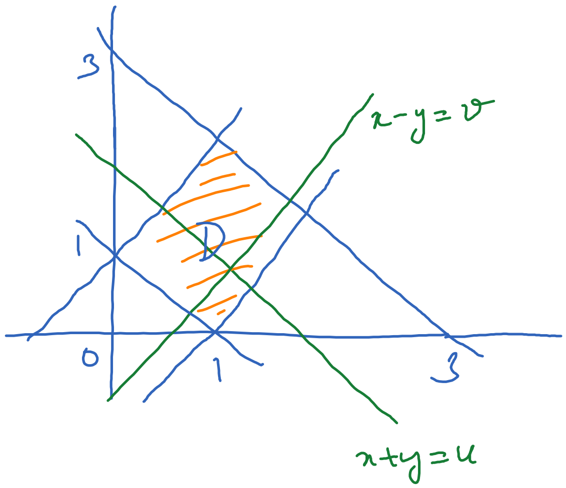
What is the stretching factor?

$$\underbrace{J = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}}_{\text{Jacobian matrix}}$$

The stretching factor is $|\det J|$.

$\det J = \dfrac{\partial(x,y)}{\partial(u,v)}$ is called the Jacobian of the change of variables.

Ex



$$\iint_D x\,dA$$

Change of variables:

$$\begin{cases} x+y=u \\ x-y=v \end{cases} \qquad \begin{matrix} 1 \le u \le 3 \\ -1 \le v \le 1 \end{matrix}$$

$$D' = [1,3]\times[-1,1]$$

$$\longrightarrow \begin{cases} x = \dfrac{u+v}{2} \\ y = \dfrac{u-v}{2} \end{cases}$$

$$J = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}, \qquad \det J = -\frac{1}{2}.$$

$$\iint_D x\,dx\,dy = \iint_{D'} \frac{u+v}{2}\left|-\frac{1}{2}\right| du\,dv = \int_{-1}^{1}\int_{1}^{3} \frac{u+v}{4}\,du\,dv = \cdots$$

Use Mathematica to draw the region D':

R = ImplicitRegion[1 ≤ x+y ≤ 3, −1 ≤ x−y ≤ 1, {x,y}]

ParametricPlot[{x+y, x−y}, {x,y} ∈ R].

Ex:

$$\iint_D (x+y)\,dA = ?$$

Using the change of variables

$$u = \frac{y}{x}, \quad v = x+y.$$

* Recall polar coords:

$$(x,y) \longrightarrow (r,\theta)$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad J = \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Stretching factor $= |\det J| = r$.

$$\iint_D \cdots \, dx\,dy = \iint_{D'} \cdots \, r\,dr\,d\theta.$$

* If $f = f(x,y,z)$ and

$$x = x(u,v,w)$$
$$y = y(u,v,w)$$
$$z = z(u,v,w)$$

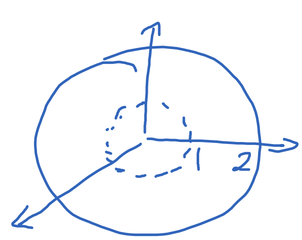
then the Jacobian matrix is

$$J = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}$$

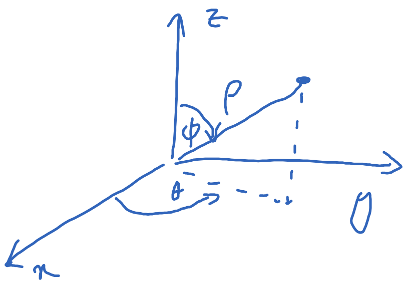
$$\det J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

stretching term $= |\det J|$.

Ex: spherical coords



$$\iint_D x^2 \, dV = ?$$



$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$J = ?$, $\det J = ?$, $|\det J| = ?$