Final exam: Some problems for review

The exam is 2 hours long and takes place in our regular classroom:

- Section 4: 2:30 PM - 4:30 PM on Tuesday 4/19/2022 at TMCB 112.
- Section 9: 11 AM - 1 PM on Saturday 4/16/2022 at TMCB 121.

It is a closed book exam, covering Sections 15.7-15.9 and 16.1-16.7. *Although you are not required to use Stokes' theorem or Divergence theorem to solve problems, you are allowed to use them if you feel inclined to.* No calculators are allowed. The following formulae will be provided on the exam:

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases}, \quad J = \rho^2 \sin\phi$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_R F(r(u,v)) \cdot (\pm r_u \times r_v) \mathrm{d}A$$

You should review methods to solve following types of problems. Also, take a look at the practice exam posted on Learning Suite.

- Find the Jacobian of a change of variables.
- Use spherical and cylindrical coordinates to evaluate triple integrals.
- Sketch by hand a vector field.
- Check if a vector field is conservative. If it is, find the potential function.
- Find curl and divergence of a vector field. Interpret them on the picture.
- Evaluate line integral using: parametrization, fundamental theorem of Calculus, Green's theorem.
- Evaluate surface integral using parametrization.

Some problems for practice:

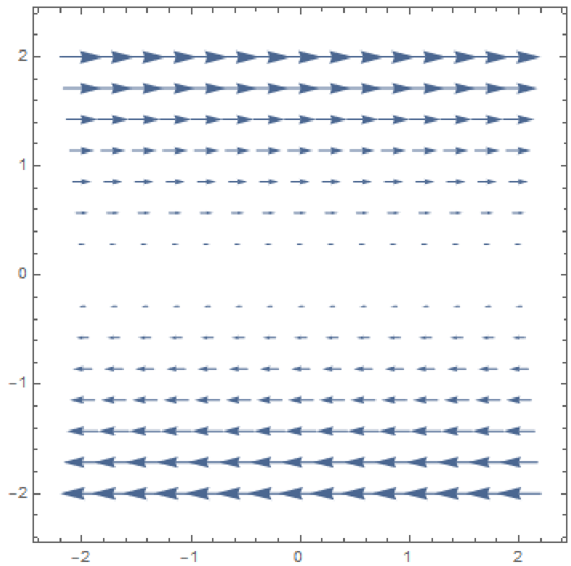
1. Evaluate the flux of the vector field $F(x,y,z) = (x+z,\, -x-y,\, 2y-4z)$ across the triangle with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,4)$ with downward orientation.
2. Evaluate the integral $\int_C (x^2+y^2)dx + (x^2-y^2)dy$ where C is the triangle with vertices at $(0,0)$, $(2,1)$, $(0,1)$ oriented in that order.
3. Parametrize the part of the parabola $y = x^2 - x$ oriented from $x = 1$ to $x = 0$.
4. Sketch the 2D vector field $F(x,y) = (y,0)$.
5. Let $F = (P,Q)$ be a vector field defined on the entire plane. What is the condition of P and Q for F to be conservative?
6. Let $F = (P,Q,R)$ be a vector field defined on the entire space. What is the condition of P,Q,R for F to be conservative?
7. Let $F = (3x^2 + 2xy,\, x^2 + 2y)$. Is F a conservative vector field? If so, what is a potential function of F?

8. Let $F = (6xy + yz,\ 3x^2 + xz,\ xy + 2z)$. Is F a conservative vector field? If so, what is a potential function of F?

9. If F is a vector field then $\operatorname{div} F$ is also a vector field. True or false?

10. If F is a vector field then $\operatorname{curl} F$ is also a vector field. True or false?

11. A 2D vector field F is visualized as follows. What can you tell about the sign of $\operatorname{div} F$ and $\operatorname{curl} F$ at point $(1, 1)$?



12. Convert the Cartesian coordinates $(x, y, z) = (-2, 2, 2\sqrt{6})$ into spherical coordinates.

13. Convert the spherical coordinates $(\rho, \theta, \phi) = (2, \pi/3, \pi/2)$ into Cartesian coordinates.

14. Describe the solid cut from the unit ball $x^2 + y^2 + z^2 \leq 1$ by the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$ in spherical coordinate. In other words, what are the ranges for ρ, θ, and ϕ ?

15. Evaluate $\iiint_E (z + 1)dV$ where E is the solid enclosed by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

16. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transform $x = u^2$, $y = v^2$.

17. Find the area of the region enclosed by the x-axis, the y-axis, and the curve $\sqrt{x} + \sqrt{y} = 1$. (*Hint: use the change of variables in the previous problem.*)

Answer keys:

1) $-2/3$
2) $\int_0^2 \int_{x/2}^1 (2x - 2y) dy dx = 0$
3) $x = 1 - t$, $y = t^2 - t$, $0 \leq t \leq 1$
4) The picture in Problem 11
5) $Q_x = P_y$ (in other words, $\operatorname{curl} F = 0$)
6) $R_y = Q_z$, $P_z = R_x$, $Q_x = P_y$ (in other words, $\operatorname{curl} F = 0$)
7) Yes, because $Q_x = P_y$. Potential function $\phi(x, y) = x^3 + x^2 y + y^2$.
8) Yes, because $\operatorname{curl} F = 0$. Potential function $\phi(x, y, z) = 3x^2 y + xyz + z^2$.
9) False
10) It depends. If F is a 3D vector field, then yes. If F is a 2D vector field, then no.
11) At $(1, 1)$, the divergence is zero, and the curl is negative.
12) $(\rho, \theta, \phi) = (4\sqrt{2}, 3\pi/4, \pi/6)$
13) $(x, y, z) = (1, \sqrt{3}, 0)$
14) $0 \leq \rho \leq 1$, $0 \leq \phi \leq \pi/6$, $0 \leq \theta \leq 2\pi$
15) 8π
16) $4uv$
17) $1/6$