# MATH 314, MIDTERM I, WINTER 2022

INSTRUCTOR: TUAN PHAM

| Name | Section # (Sec. 4: 12 - 1PM, Sec. 9: 11 - 12PM) |
|---|---|
| | |

**Instructions:**

- This is a closed-book exam, 2 hours long. No calculators are allowed.
- For Problems 1-12, use your pencil/pen to fill in the bubbles on this front page (see below).
- For Problems 13, 14, 15, make sure to show all necessary steps. Mysterious answers will receive little or no credit.
- Some formulae are provided below.
- Do not discuss the exam with anyone between Feb 2 and Feb 4.

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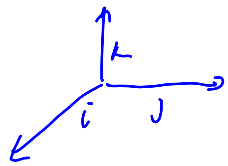
10. Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ
11. Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ
12. Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

| Problem | Possible points | Earned points |
|---|---|---|
| 1-12 | 24 | |
| 13 | 10 | |
| 14 | 10 | |
| 15 | 10 | |
| Total | 54 | |

$$\kappa = \frac{|r' \times r''|}{|r'|^3}, \quad \tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}, \quad N = \frac{T'}{|T'|}, \quad B = T \times N, \quad a_T = \frac{r' \cdot r''}{|r'|}, \quad a_N = \frac{|r' \times r''|}{|r'|}.$$

**Problem 1.** (2 points) Let $i$, $j$, $k$ be the unit vector along the $x, y, z$ axes, respectively, of a coordinate system. Which of the following is equal to $i \times k$ ?




A. $j$

B. $-j$

C. $0$

**Problem 2.** (2 points) Consider three points in the space $A(1, 0, 1)$, $B(2, 1, 0)$, $C(-1, 2, -1)$. Which of the following is equal to $(\overrightarrow{AB} \times \overrightarrow{BC}) \cdot \overrightarrow{AC}$

$\vec{AB} = \langle 1, 1, -1 \rangle$

$\vec{BC} = \langle -3, 1, -1 \rangle$

$\vec{AB} \times \vec{BC} = \langle 0, 4, 4 \rangle$

$\vec{AC} = \langle -2, 2, -2 \rangle$

A. 0

B. 1

C. 8

D. 16

**Problem 3.** (2 points) To check if four points $A, B, C, D$ lie on the same plane, which of the following methods is correct?

A. Check if $\overrightarrow{AB} \times \overrightarrow{CD} = 0$

B. Check if $\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$

C. Check if $(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = 0$

D. Check if $(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot (\overrightarrow{BC} \times \overrightarrow{BD}) = 0$

**Problem 4.** (2 points) Which of the following is the correct parametrization of the line passing through point $(1, 2, -1)$ and parallel to the line $x = 2t + 1$, $y = t$, $z = t - 3$.

A. $r(t) = \langle t + 1,\ 2,\ -3t - 1 \rangle$

B. $r(t) = \langle 2t + 1,\ t + 2,\ t - 1 \rangle$

C. $r(t) = \langle t + 2,\ 2t + 1,\ -t + 1 \rangle$

D. $r(t) = \langle 2t + 1,\ t,\ t - 3 \rangle$

**Problem 5.** (2 points) $z = x^2 - 4y^2$ is an equation of which of the following surfaces?

A. ellipsoid
B. cone
C. elliptic paraboloid
D. hyperbolic paraboloid

**Problem 6.** (2 points) Consider a triangle $ABC$ with vertices at $A(1,1,0)$, $B(2,0,0)$, $C(2,2,-2)$. What is the angle of the triangle at vertex $B$ ?

A. 30°
B. 45°
C. 60°
D. 90°

$\vec{BA} = \langle -1, 1, 0\rangle \rightsquigarrow |\vec{BA}| = \sqrt{2}$

$\vec{BC} = \langle 0, 2, -2\rangle \rightsquigarrow |\vec{BC}| = \sqrt{8}$

$\vec{BA} \cdot \vec{BC} = 2$

$\cos\theta = \dfrac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}||\vec{BC}|} = \dfrac{2}{\sqrt{2}\sqrt{8}} = \dfrac{1}{2}$

$\rightsquigarrow \theta = 60°$

**Problem 7.** (2 points) The volume of the parallelepiped formed by three vectors $a, b, c$ is equal to
A. $a \cdot (b \times c)$
B. $|a \cdot (b \times c)|$
C. $|b \cdot (a \times c)|$
D. Only B and C are correct.
E. A, B, C are all correct.

**Problem 8.** (2 points) Choose the correct statement.

A. The velocity is always tangent to the trajectory.
B. The acceleration is always perpendicular to the velocity.
C. The acceleration is always perpendicular to the trajectory.
D. The acceleration is the derivative of the speed.

**Problem 9.** (2 points) Which of the following statements is true about a smooth curve on a plane?

A. The curvature is always zero.
B. The torsion is always zero.
C. The curvature is always nonzero.
D. The torsion is always nonzero.

**Problem 10.** (2 points) Let $r(t) = \langle t, e^t, 1\rangle$. Which of the following is the correct value of the integral $\int_0^1 r(0) \cdot r'(t)dt$ ?

A. 0
B. 1
C. $e$
D. $e - 1$

$$\left.\begin{aligned} r(0) &= \langle 0, 1, 1\rangle \\ r'(t) &= \langle 1, e^t, 0\rangle \end{aligned}\right\} \quad r(0)\cdot r'(t) = e^t$$

$$\int_0^1 e^t\,dt = e^t\Big|_0^1 = e^1 - e^0 = e - 1$$

**Problem 11.** (2 points) Determine the limit

$$\lim_{t\to\pi}\frac{1}{t-\pi}\langle \sin t,\, t^2 - t\pi, 0\rangle = \lim\left\langle \frac{\sin t}{t-\pi},\ \frac{t^2-t\pi}{t-\pi},\ \frac{0}{t-\pi}\right\rangle$$

$$= \lim_{t\to\pi}\left\langle \underbrace{\frac{\sin t}{t-\pi}}_{\downarrow -1},\ \underbrace{t}_{\downarrow \pi},\ \underbrace{0}_{\downarrow 0}\right\rangle$$

by L'Hospital rule

A. $\langle -1, \pi, 0\rangle$
B. $\langle 1, \pi, 0\rangle$
C. $\langle 0, 0, 0\rangle$
D. does not exist

**Problem 12.** (2 points) The intersection of the cylinder $x^2+4y^2 = 4$ and the ellipsoid $x^2+4y^2+z^2 = 5$ *above* the $xy$-plane has a parametrization

A. $r(t) = \langle 2\sin t,\, \cos t,\, -1\rangle$
B. $r(t) = \langle \sin t,\, 2\cos t,\, 1\rangle$
C. $r(t) = \langle \cos t,\, 2\sin t,\, \sqrt{5}\rangle$
D. $r(t) = \langle 2\cos t,\, \sin t,\, 1\rangle$

$$\begin{cases} x^2+4y^2=4 \\ x^2+4y^2+z^2=5 \end{cases} \rightsquigarrow \begin{cases} x^2+4y^2=4 \\ z^2=1 \end{cases} \rightsquigarrow z=1$$

(because $z>0$)

$$\left(\frac{x}{2}\right)^2 + y^2 = 1 \rightsquigarrow \begin{cases} x/2 = \cos t \\ y = \sin t \end{cases} \rightsquigarrow \begin{cases} x = 2\cos t \\ y = \sin t \end{cases}$$

**Problem 13.** (10 points) Find the equation of the line tangent to the curve $r(t) = \langle t, t-1, e^t \rangle$ at the point $(0, -1, 1)$.

$r'(t) = \langle 1, 1, e^t \rangle$

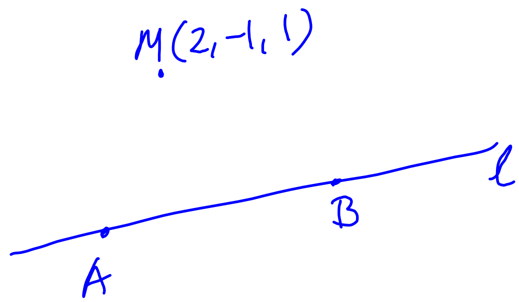
At the point $(0,-1,1)$, $t = 0$.

$r'(0) = \langle 1,1,1 \rangle$ ; direction vector of the curve at $(0,-1,1)$.

Eq. of the tangent line:

$$\begin{cases} x = t \\ y = -1 + t \\ z = 1 + t \end{cases}$$

**Problem 14.** (10 points) Find the equation of the plane that passes through point $(2,-1,1)$ and contains the line $x = 1 - t$, $y = 2 - 3t$, $z = 1 + t$.




Take two points on the line:

$t=0$: $A(1,2,1)$

$t=1$: $B(0,-1,2)$

$$\vec{AM} = \langle 1, -3, 0 \rangle$$

$$\vec{AB} = \langle -1, -3, 1 \rangle$$

normal vector of the plane:

$$\vec{AM} \times \vec{AB} = \langle -3, -1, -6 \rangle$$

Eq. of the plane:

$$-3(x-1) + (-1)(y-2) + (-6)(z-1) = 0$$

$$\rightsquigarrow \quad -3x - y - 6z + 3 + 2 + 6 = 0$$

$$\rightsquigarrow \quad 3x + y + 6z = 11$$

**Problem 15.** (10 points) Find the torsion of the helix $r(t) = \langle \cos t, \sin t, at \rangle$, where $a$ is a constant.

$$r'(t) = \langle -\sin t, \cos t, a \rangle$$

$$r''(t) = \langle -\cos t, -\sin t, 0 \rangle$$

$$r'''(t) = \langle \sin t, -\cos t, 0 \rangle$$

$$r' \times r'' = \langle a \sin t, -a \cos t, 1 \rangle$$

$$(r' \times r'') \cdot r''' = a \sin^2 t + a \cos^2 t + 0 = a$$

$$|r'| = \sqrt{\sin^2 t + \cos^2 t + a^2} = \sqrt{1 + a^2}$$

Torsion

$$\tau = \frac{(r' \times r'') \cdot r'''}{|r'|^2} = \frac{a}{1 + a^2}$$