Stochastic cascade solutions of the Navier-Stokes equations

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Diffusion equation – Probabilistic representation

In $\mathbb{R}^d \times (0, \infty)$, consider the initial-value problem

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u & = & 0, \\ u(x, 0) & = & u_0(x). \end{cases}$$

Classical solution:

$$u(x, t) = \int_{\mathbb{R}^d} \underbrace{\frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|y - x|^2}{2t}\right)}_{\Phi(y-x,t)} u_0(y) dy.$$

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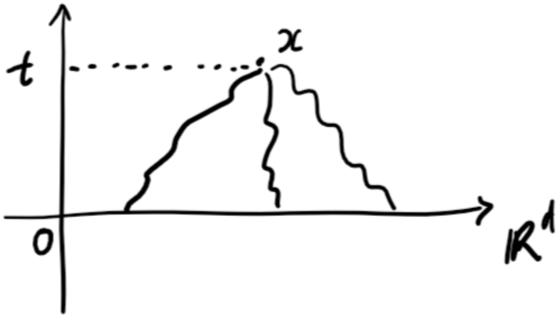
Observe: $\Phi(\cdot - x, t)$ is the p.d.f of an $\mathcal{N}(x, t\, I_d)$-random variable in $\mathbb{R}^d$, e.g. Brownian motion B_t^x.

$$u(x, t) = \mathbb{E}[u_0(B_t^x)].$$

Diffusion equation – Probabilistic representation



"Stochastic characteristic curves"

Reaction-Diffusion equation – Probabilistic representation

$$\begin{cases} \partial_t u - \frac{1}{2}\Delta u &= -K(x)u, \\ u(x,0) &= u_0(x). \end{cases}$$

Feynman-Kac formula (1940s):

$$u(x,t) = \mathbb{E}\left[u_0(B_t^x)\exp\left(-\int_0^t K(B_s^x)ds\right)\right].$$

The problem can be formulated and generalized (with drift term ∇u and forcing f) by Itô calculus (1950s).

KPP-Fisher equation

In $\mathbb{R} \times (0, \infty)$, consider the equation (Kolmogorov-Petrovskii-Piskunov (KPP), Fisher, 1937):

$$\begin{cases} u_t - \frac{1}{2}u_{xx} & = & u^2 - u, \\ u(x,0) & = & u_0(x). \end{cases}$$

With $\Psi = e^{-t}\Phi$,

$$u(x,t) = \int_{\mathbb{R}} \Psi(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}} \Psi(x-y,s)u^2(y,t-s)dyds.$$

Noting that Ψ is a p.d.f on $\mathbb{R} \times (0, \infty)$, McKean (1975) gave a probabilistic description of this equation by branching process.

KPP-Fisher equation

$T \sim Exp(1)$: holding time (the clock).

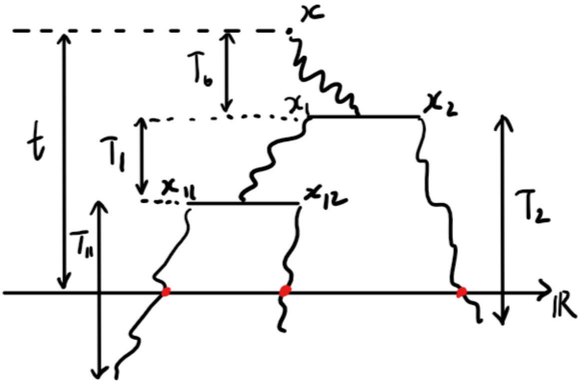
$$u(x,t) = \mathbb{E}\left[u_0(B_t^x)\mathbb{1}_{T>t}\right] + \mathbb{E}\left[u^2(B_T^x, t-T)\mathbb{1}_{T\le t}\right]$$

In other words, $u(x,t) = \mathbb{E}[\mathbf{X}(x,t)]$ where

$$\mathbf{X}(x,t) = \begin{cases} u_0(B_t^x) & \text{if} \quad T > t, \\ \mathbf{X}^{(1)}(B_T^x, t-T)\,\mathbf{X}^{(2)}(B_T^x, t-T) & \text{if} \quad T \le t. \end{cases}$$

Here $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are i.i.d copies of $\mathbf{X}$ and are independent of T.

KPP-Fisher equation – Branching process



$$\mathbf{X}(x,t) = u_0(B^{x_{11}}_{t-T_0-T_1})u_0(B^{x_{12}}_{t-T_0-T_1})u_0(B^{x_2}_{t-T_0}).$$

Diffusion-reaction equation – Fourier domain

The heat operator $\partial_t - \Delta$ naturally induces a clock in the Fourier domain. For example,

$$u_t - u_{xx} = bu, \quad u(x,0) = u_0(x).$$

In Fourier domain,

$$\hat{u}(\xi,t) = e^{-t\xi^2}\hat{u}_0(\xi) + \int_0^t e^{-s\xi^2} b\hat{u}(\xi, t-s)ds.$$

Put $\chi = \xi^2\hat{u}$. Then

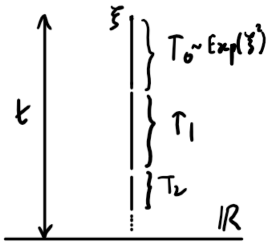
$$\chi(\xi,t) = e^{-t\xi^2}\chi_0(\xi) + \int_0^t \underbrace{\xi^2 e^{-s\xi^2}}_{\text{p.d.f}} \frac{b}{\xi^2}\chi(\xi, t-s)ds.$$

Diffusion-reaction equation – Fourier domain

$$\chi(\xi, t) = \mathbb{E}[\mathbf{X}(\xi, t)]$$

where

$$\mathbf{X}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T > t, \\ \frac{b}{\xi^2}\mathbf{X}(\xi, t - T) & \text{if} \quad T \leq t. \end{cases}$$



$$\mathbf{X}(\xi, t) = \left(\frac{b}{\xi^2}\right)^{N_t} \chi_0(\xi), \quad N_t = \inf\{n : T_0 + T_1 + ... + T_n > t\}$$

Diffusion-reaction equation – Fourier domain

$$u_t - u_{xx} = (\cos x)u, \quad u(x,0) = u_0(x).$$

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-t\xi^2} + \frac{c}{2}\int_0^t \xi^2 e^{-s\xi^2}\left(\frac{\hat{u}(\xi-1,t-s)}{\xi^2} + \frac{\hat{u}(\xi+1,t-s)}{\xi^2}\right)ds$$

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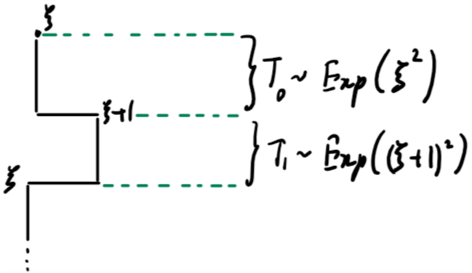

$$\mathbf{X}(\xi,t) = \begin{cases} \hat{u}_0(\xi) & \text{if} \quad T > t, \\ \frac{c}{\xi^2}\mathbf{X}(W, t-T) & \text{if} \quad T \le t. \end{cases}$$

$$\mathbb{P}_\xi(W = \xi - 1) = \mathbb{P}_\xi(W = \xi + 1) = 1/2.$$

Navier-Stokes equations

$$(\mathrm{NSE}): \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ \mathrm{div}\ u = 0 & \text{in} \quad \mathbb{R}^d \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^d. \end{cases}$$

Integro-differential equation:

$$u(x,t) = e^{\Delta t} u_0 - \int_0^t e^{\Delta s} \mathbf{P} \mathrm{div}[u(t-s) \otimes u(t-s)] ds.$$

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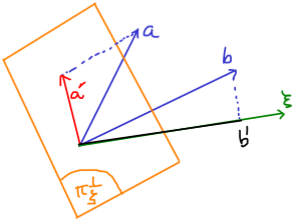
$$\hat{u}(\xi,t) = e^{-|\xi|^2 t} \hat{u}_0(\xi) + c_0 \int_0^t e^{-|\xi|^2 s} |\xi| \int_{\mathbb{R}^d} \hat{u}(\eta, t-s) \odot_\xi \hat{u}(\xi - \eta, t-s) d\eta ds$$

where $a \odot_\xi b = -i(e_\xi \cdot b)(\pi_{\xi^\perp} a)$.



$$a \odot_\xi b = -i\, b' a'$$

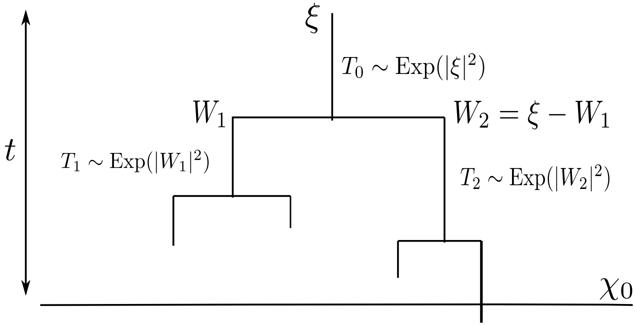
Normalization to (FNS): LJS 1997, Bhattacharya et al 2003.

$$\begin{aligned}\chi(\xi,t) &= e^{-t|\xi|^2}\chi_0(\xi)\\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2\int_{\mathbb{R}^d}\chi(\eta,t-s)\odot_\xi\chi(\xi-\eta,t-s)H(\eta|\xi)\,d\eta\, ds\end{aligned}$$

where $\chi = c_0\hat{u}/h$ and $H(\eta|\xi) = \frac{h(\eta)h(\xi-\eta)}{|\xi|h(\xi)}$.

h: *majorizing kernel*, i.e. $h * h = |\xi| h$.

Cascade structure of FNS

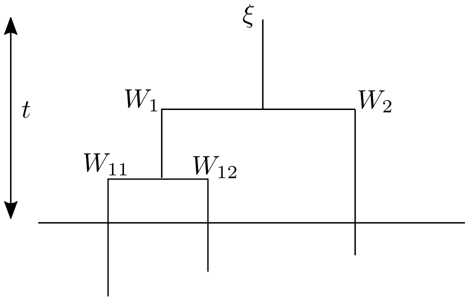


Define a stochastic multiplicative functional recursively as

$$\mathbf{X}_{\mathrm{FNS}}(\xi, t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 > t, \\ \mathbf{X}^{(1)}_{\mathrm{FNS}}(W_1, t - T_0) \odot_\xi \mathbf{X}^{(2)}_{\mathrm{FNS}}(\xi - W_1, t - T_0) & \text{if} \quad T_0 \leq t. \end{cases}$$

An example of $\mathbf{X}_{\mathrm{FNS}}$

Consider the following event:



On this event,

$$\mathbf{X}_{\mathrm{FNS}}(\xi, t) = (\chi_0(W_{11}) \odot_{W_1} \chi_0(W_{12})) \odot_\xi \chi_0(W_2).$$

Three ingredients: clocks, branching process, product.
Cascade structure = clocks + branching process.

Stochastic cascade solutions – Two issues

We referred to solutions given by the expectation of a multiplicative stochastic functional $\mathbf{X}$ as *stochastic cascade solutions*.

Cascade structure $=$ clocks $+$ branching process.

There are two potential issues with this construction:

- Stochastic explosion: the branching process may never stop, making $\mathbf{X}$ not well-defined.
- Existence of the expectation: it may happen that $\mathbb{E}|\mathbf{X}| = \infty$.

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Existence of expectation $\longleftrightarrow$ smallness of initial condition.

Stochastic explosion: (1) *Can it happen?* (2) *Any connection with non-uniqueness?*

Explosion

Branching process may never stop, potentially making $\mathbf{X}_{FNS}$ not well-defined.

- Property of cascade structure, not of product.
- Depending on the majorizing kernel h.
- 3D self-similar cascade $h_{\mathrm{dilog}}(\xi) = C|\xi|^{-2}$: stochastic explosion a.s. (Dascaliuc, Pham, Thomann, Waymire 2019)
- 3D Bessel cascade $h_{\mathrm{b}}(\xi) = C|\xi|^{-1}e^{-|\xi|}$: non-explosive a.s. (Orum, Pham 2019)

What if explosion happens?

If explosion happens (e.g. the self-similar cascade), how can we define a stochastic cascade solution?

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Introducing a *ground state*:

$$\mathbf{X}_0(\xi, t) \equiv 0,$$
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Ignore the product and regard $\mathbf{X}_n$ as a scalar:

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Probability of non-explosion

- For $\chi_0 \equiv 1$ and $\mathbf{X}_0 \equiv 1$ then $\mathbf{X}_n = 1$ for all n. Thus, $\chi = \lim \mathbb{E}\mathbf{X}_n = 1$.
- For $\chi_0 \equiv 1$ and $\mathbf{X}_0 \equiv 0$ then $\chi = \lim \mathbb{E}\mathbf{X}_n = \mathbb{P}(S_\xi > t)$. where S_ξ is the shortest branch of the tree rooted at ξ.
- Both of functions χ above solve the equation

$$\begin{aligned}\chi(\xi,t) &= e^{-t|\xi|^2}\chi_0(\xi) \\ &+ \int_0^t e^{-s|\xi|^2}|\xi|^2 \int_{\mathbb{R}^d} \chi(\eta,t-s)\chi(\xi-\eta,t-s)H(\eta|\xi)d\eta ds\end{aligned}$$

- In physical domain: $\partial_t u - \Delta u = \sqrt{-\Delta}(u^2)$.

Explosion and non-uniqueness

Dascaliuc, Thomann, Michalowski, Waymire (2015)

Explosion is equivalent to non-uniqueness of a scalar pseudo-differential equation

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known as the *genealogical Navier-Stokes equation*.

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known as the *genealogical Navier-Stokes equation*.

- This equation has at least two solutions when $h = c/|\xi|^2$: one is time-independent, the other is time-dependent.
- Hyper-explosion, e.g. the α-Riccati equation $u' = -u + u^2(\alpha t)$, helps us construct more than two solutions.
- Does explosion provide a pathway to non-uniqueness of the Navier-Stokes equations?

Thank You!