Research Statement

Tuan N. Pham

I am interested in partial differential equations, stochastic processes, fluid dynamics and numerical analysis. The main theme of my research is the regularity of the Navier-Stokes equations:

$$\text{(NSE)}: \begin{cases} u_t - \Delta u + u\nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty), \\ u(x,0) = u_0(x). \end{cases}$$

The global well-posedness of the three-dimensional NSE—a Millennium problem—is one of the most fundamental problems in mathematical fluid mechanics. My approaches to this challenging problem are analytic and probabilistic, and are directed toward answering the following questions: **(1)** Under what conditions does a solution remain smooth for all time? Such a condition could be *a priori* (involving the initial data) or *a posteriori* (based on a numerical approximation of the solution). **(2)** What roles does the flow's physical boundary play in the formation of singularities? **(3)** What do NSE look like at a stochastic scale (in the analogy to Brownian motions in the context of diffusion equations)? **(4)** How may such a stochastic mechanism be useful in the regularity theory and numerical simulation?

Regarding question 1, the less information about the approximate solution required by an *a posteriori* regularity criterion the better. The previously available *a posteriori* criteria either require the knowledge of the derivatives (in addition to the magnitude) of the approximate solution or are not invariant under the natural scaling of NSE. In [29], I derived a criterion which overcomes these two issues. *Question 2* is the motivation of my PhD thesis [28]. My study of the boundary regularity results in a new application of pressure decomposition introduced by Seregin [34] to show the stability of singularities near the boundary. I became interested in *question 3* through the groundbreaking work of Le Jan and Sznitman (LJS) [18], and through my collaboration with Radu Dascaliuc, Enrique Thomann, and Edward Waymire at Oregon State University. In LJS's method, a solution is represented as the expected value of a branching process on trees. In their construction of solutions, an artificial coin-tossing device was needed to terminate the branching process. Together with my collaborators, we introduce a simple method, intrinsic to the nature of the branching process, to construct a solution without coin-tossing [8]. Inspired by LJS's stochastic model for NSE, we introduce a new class of probabilistic models called *doubly stochastic Yule cascades* [9,10] which turns out to be a natural generalization of probabilistic models in data compression, percolation, aging, and cancer growth. *Regarding question 4*, in [26] I identify a symmetric property of the Fourier-transformed NSE from the *conservation of frequencies*—a property at the stochastic scale—and use it to obtain global strong solutions for initial data whose Fourier transforms are supported on the half-space (no smallness condition is required).

Below, I will describe in more details my featured research works—their methodology and their connection with the aforementioned questions.

1 A posteriori regularity criterion

Consider a sequence of approximate solutions $\{u_\varepsilon\}_{\varepsilon>0}$ that converges to the true solution u in some sense. One can think of, for example, a sequence of approximate solutions coming from numerical simulations, or from perturbing the initial data, or from Leray's mollification $[u\nabla u]_\varepsilon$. Full information about the behavior of this sequence as $\varepsilon \downarrow 0$ would give useful information about the exact strong solution. In practice, however, we only have information about finitely many approximate solutions. Let us assume that we know *only one* approximate solution for a certain value of ε. The question is: *how large can ε be so that we can infer the global existence of the exact strong solution from u_ε?* From a practical perspective, the larger ε is the better.

A global regularity criterion of this kind is known in literature as an *a posteriori* regularity criterion, which serves as a check for an approximate/numerical solution to guarantee the existence of the exact solution.

The previously available *a posteriori* regularity criteria are not invariant under the natural scaling of NSE or require information on the derivatives of the approximate/numerical solution (see [5, 12, 20, 24, 25]). This observation motivates my work [29], in which I give a simple scaling-invariant posteriori regularity criterion that only requires the knowledge of the L^∞-norm of the approximate solution. A simplified statement of the main result is as follows.

Theorem 1 **([29])** *Consider the mollified Navier-Stokes system*

$$(\mathrm{NSE})_\varepsilon : \begin{cases} u_t - \Delta u + [u\nabla u]_\varepsilon + \nabla p = 0, \quad \operatorname{div} u = 0, \\ u(x,0) = u_0(x). \end{cases}$$

Suppose $(\mathrm{NSE})_\varepsilon$ *has a global strong solution* u_ε *bounded by* M*. If* $\varepsilon \lesssim M^{-1}\exp(-\|u_0\|_{L^2}^4 M^2)$ *then* NSE *has a global strong solution bounded by* $2M$*.*

At a local scale, the proof is a nice application of the ϵ-regularity criterion introduced by Caffarelli-Kohn-Nirenberg (1982). A rate of $\varepsilon \lesssim M^{-1}$ would be more desirable for practical purposes. However, the time-dependence nature complicates the problem. Larger total energy naturally requires finer resolution in order to capture microscopic structures of the exact solution. Therefore, the presence of the total energy in the estimate of ε seems inevitable.

2 On Le Jan–Sznitman's stochastic approach

Le Jan and Sznitman (1997) introduced an elegant stochastic approach to the deterministic NSE in which they expressed the function $\chi = c_0\hat{u}/h$ as the expected value of a random variable: $\chi = \mathbb{E}_\xi \mathbf{X}$. Here, c_0 is a universal constant and h is a function satisfying $h*h = |\xi|h$, called a *standard majorizing kernel.* Such kernels include the Bessel kernel $h_{\mathrm{b}}(\xi) = ce^{-|\xi|}|\xi|^{-1}$ for $d = 3$, and the scale-invariant kernel $h_{\mathrm{in}}(\xi) = c_d|\xi|^{1-d}$ for $d \geq 3$. The random variable $\mathbf{X}$ is defined implicitly as

$$\mathbf{X}(\xi,t) = \begin{cases} \chi_0(\xi) & \text{if} \quad T_0 \geq t, \\ \mathbf{X}^{(1)}(W_1, t-T_0)\odot_\xi \mathbf{X}^{(2)}(W_2, t-T_0) & \text{if} \quad T_0 < t \end{cases} \tag{1}$$

where the "circle-dot" product is a non-commutative, non-associative vector operation encoding the divergence-free condition of NSE, T_0 is an exponential clock with mean $|\xi|^{-2}$, $W_1 \in \mathbb{R}^d$ and $W_2 = \xi - W_1$ are random variables with a distribution determined by h, and $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are two independent copies of $\mathbf{X}$. Applying

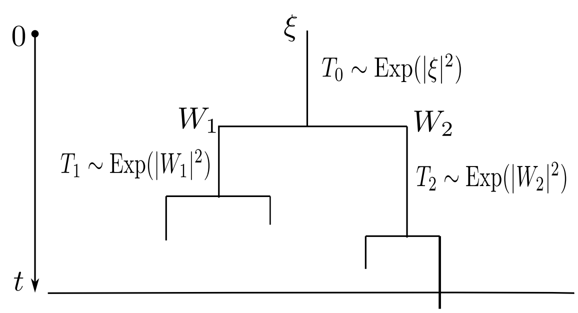

Figure 1: Cascade figure illustrating the branching process. The recursive formula (1) arrives at a closed form when all paths of the tree cross the time horizon t.

the formula (1) repeatedly, one may realize that $\mathbf{X}$ is defined on a branching process (see Figure 1 for a cascade figure illustrating this branching process). In closed form, $\mathbf{X}$ is a circle-dot product of χ_0 evaluated at random locations in space.

This framework provides a probabilistic perspective for the well-posedness problem of the Navier-Stokes equations and has led to several interesting results which I will describe below.

Conservation of frequencies

The Navier-Stokes equations have a natural scaling symmetry: $u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$, $p(x,t) \to \lambda p(\lambda x, \lambda^2 t)$. There is another symmetry property at the stochastic scale:

$$\chi_0 \to e^{\xi\cdot a}\chi_0, \quad \mathbf{X} \to e^{\xi\cdot a}\mathbf{X} \quad (a \in \mathbb{R}^d) \tag{2}$$

which comes from the observation that after each branching, the sum of all the wave numbers at the leaves of the tree in Figure 1 is always equal to the starting wave number ξ. This conservation of frequencies was noted in [7] by Dascaliuc et al. although the symmetry (2) was not identified there. At the deterministic scale, the conservation of frequencies manifests in the symmetry:

$$\hat{u}_0 \to e^{\xi\cdot a}\hat{u}_0, \quad \hat{u} \to e^{\xi\cdot a}\hat{u} \quad (a \in \mathbb{R}^d). \tag{3}$$

An interesting application to the global well-posedness of NSE immediately follows [26]: if $\hat{u}_0$ is supported in a half-space of $\mathbb{R}^d$ (but can be large), then one can select a suitable vector $a \in \mathbb{R}^d$ so that $\hat{u}_{0a} = e^{\xi\cdot a}\hat{u}_0$ is sufficiently small in some scale-critical space; then NSE with initial data u_{0a} has a global solution u_a; by reversing the scaling, one obtains a global solution $u = \mathcal{F}^{-1}\{e^{-\xi\cdot a}\hat{u}_a\}$ corresponding to the initial data u_0.

This simple argument needs $\hat{u}_0$ to be supported in a half-space, which excludes all real-valued initial data u_0 of NSE. Nevertheless, it at least shows that the global well-posedness of NSE is difficult because the initial velocity field must be real! In addition, the argument simplifies and generalizes the global regularity criteria in [13, 22]. A result in my upcoming paper [26] is as follows.

Theorem 2 ([26]) *If supp* $\hat{u}_0 \subset \mathbb{R}^d_+$ *and* $\hat{u}_0 \in L^2$, *where* $d = 2$ *or 3, then the strong solution is global. If supp* $\hat{u}_0 \subset \mathbb{R}^d_+$ *and* $\hat{u}_0 \in L^p$, *where* $1 \le p < \frac{d}{d-1}$, *then the strong solution is global.*

Nonuniqueness and blowup of the Montgomery-Smith equation

There are intrinsic connections between the stochastic cascade illustrated in Figure 1 and the equation

$$\text{(MS)}: \; u_t - \Delta u = \sqrt{-\Delta}(u^2), \quad u(x,0) = u_0(x)$$

introduced by Montgomery-Smith [23] for a study of the finite-time blowup of the Navier-Stokes equations. If we denote by $\rho(\xi,t)$ the probability that a cascade starting at a wave number $\xi \in \mathbb{R}^d$ crosses the horizon t after finitely many branchings, then the function $v = \mathcal{F}^{-1}\{h\rho/c_0\}$ solves MS. This connection makes MS a natural equation to study the stochastic structure of the NSE. In particular, the uniqueness of solutions to MS implies the non-explosion of the cascade (i.e. the probability of non-explosion, $\rho(\xi,t)$, is equal to 1). Vice versa, the explosion of the cascade implies the nonuniqueness of solutions to MS. Together with my collaborators Radu Dascaliuc, Enrique Thomann, and Edward Waymire, we achieve several interesting results in this direction.

Theorem 3 ([8]) *For* $d = 3$, *we have the following statements.*

(i) For $u_0(x) = \frac{2}{\pi}\frac{1}{|x|}$, MS *has at least two solutions: the time-independent solution* $u_1 = u_0$ *and the time-decaying solution* $u_2 = (\frac{2}{\pi})^{3/2}\mathcal{F}^{-1}\{|\xi|^{-2}\rho(\xi,t)\}$.

(ii) Consider the initial data of the form $u_0 = 2a(1+|x|^2)^{-1}$. *If* $a \in [0,1)$, MS *has a unique global solution in* $L^5(\mathbb{R}^3 \times (0,\infty))$. *If* $a = 1$, MS *has a time-independent solution and the solution is unique. If* $a > 1$, MS *has a unique solution which escapes* $L^5(\mathbb{R}^3 \times (0,T))$ *for some* $T < \infty$.

The nonuniqueness result in Part (i) is worth noting. It is conjectured by Jia and Šverák [17] that uniqueness of scale-invariant solutions to NSE may break for large scale-invariant initial data. Although the conjecture has not been settled for NSE, our result shows that it has an affirmative answer for the toy model MS.

Doubly Stochastic Yule cascades and the stochastic explosion

Inspired by Le Jan and Sznitman's stochastic method for NSE, we introduce a new class of probabilistic models called *Doubly Stochastic Yule* (*DSY*) *cascades*. In literature, the classical Yule process is a pure birth

Markov process with (constant) rate $\lambda > 0$ and can be viewed as a tree-indexed family $\{\lambda^{-1}T_v\}_{v\in\mathbb{T}}$ where $\mathbb{T} = \{0\} \cup \cup_{n=1}^{\infty}\{1,2\}^n$ is a rooted full binary tree and $\{T_v\}_{v\in\mathbb{T}}$ is a family of i.i.d. mean-one exponential random variables (called *clocks*, or the *standard Yule cascade*). As in the case of doubly stochastic Poisson process, one may allow the intensities to be positive random variables dependent on the vertices but independent of the clocks. This defines a DSY cascade $\{\lambda_v^{-1}T_v\}_{v\in\mathbb{T}}$.

DSY cascades arise naturally from the perspective of evolutionary differential equations, including the Navier-Stokes equations and the Fisher-KPP equation, to purely probabilistic models of stochastic phenomena, such as percolation and aging models. The special case $\lambda_v = \alpha^{-|v|}$ is interpreted in terms of data compression and percolation [3]. It has also been considered for important cellular biology questions related to aging and cancer, where generational cell division rates decrease with generations [4].

Stochastic explosion—the situation when the tree in Figure 1 produces infinitely many branches before a time horizon—is an intrinsic issue in the Le Jan-Sznitman's construction of solutions to NSE. In a broader sense, stochastic explosion is a natural probabilistic problem associated with any DSY cascade. The DSY cascade associated with NSE has a branching Markov chain structure underlying the intensities $\{\lambda_v\}_{v\in\mathbb{T}}$. Our study of DSY cascades with this underlying structure has been quite fruitful. In [9], we give criteria for the non-explosion of Markov-type DSY cascades using the technique of large deviations. In [10], we use a new probabilistic technique called "cutset arguments" and a greedy algorithm to respectively establish non-explosion and explosion criteria. Notable applications include the following.

Theorem 4 ([9, 10]) *The Bessel cascade (corresponding to $h = h_{\mathrm{b}}$) of the 3-dimensional* NSE *is a.s. non-explosive. The self-similar cascade (corresponding to $h = h_{\mathrm{in}}$) of the 3-dimensional* NSE *is a.s. explosive. The self-similar cascade of the d-dimensional* NSE*, for $d \geq 12$, is a.s. non-explosive.*

Using a tree-partitioning technique to compare a DSY cascade with the standard Yule cascade, I obtain the following non-explosion criterion (reminiscent of Feller's non-explosion criterion for Markov processes and Pemantle-Peres's non-explosion criterion for percolation on trees).

Theorem 5 ([27]) *Let $\{\lambda_v^{-1}T_v\}_{v\in\mathbb{T}}$ be a DSY cascade with deterministic intensities which only depend on the generational heights, i.e. $\lambda_v = \lambda_{|v|}$. Suppose the sequence $\{\lambda_n\}$ is nondecreasing. Then the cascade is a.s. non-explosive if and only if $\sum \frac{1}{\lambda_n} = \infty$.*

3 Minimal blowup data in the presence of boundaries

Assuming that there exist initial data leading to finite-time singularities, one might wonder if the *minimum* (in a certain norm) among them exists. From a control theory perspective, the size of the minimal blowup data represents the minimal cost to generate a blowup solution. For NSE in the whole space $\mathbb{R}^3$, the answer is affirmative in various settings of the initial data [14–16, 21, 30, 32]. In my PhD thesis [28], I investigate the influence of the physical boundary on the existence of minimal blowup data. The main difficulties are (1) the low regularity of pressure at the boundary, and (2) the instability of singularities with respect to localization of domains.

To deal with the instability issue, the force term is incorporated in the equation to make the data size more stable under the change of domains and perturbation of the equation. Let NSE_+ be the forced Navier-Stokes problem in $\mathbb{R}^3_+$ with non-slip boundary condition. Denote by $\rho^+_{\max}$ the supremum of all $\rho > 0$ such that NSE_+ is globally well-posed for every initial data u_0 and force f satisfying $\|(u_0, f)\|_{X\times Y} = \|u_0\|_X + \|f\|_Y < \rho$. One can define $\rho_{\max}$ the same way but for the whole space. To deal with the the pressure regularity at the boundary, I work with the class of weak solutions introduced by Seregin and Šverák [33], which are suitable for the decomposition of the pressure near the boundary. Put

$$Y_q = \{f : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}^3 : \ t^{q_*} f \in L^q(\mathbb{R}^3 \times (0,\infty))\}$$

with $\|f\|_{Y_q} = \|t^{q_*} f\|_{L^q}$ and $q_* = 3/2 - 5/(2q)$. I obtain the following.

Theorem 6 ([28]) *Let $X = \{0\}$ and $Y = Y_q$ for $5/2 < q < 3$. Then $\rho^+_{\max} \leq \rho_{\max}$.* NSE *has a minimal blowup data if $\rho_{\max} < \infty$.* NSE_+ *has a minimal blowup data if $\rho^+_{\max} < \rho_{\max}$.*

When $\rho^+_{\max} < \rho_{\max}$, the boundary facilitates blowup in the sense that all singularities, if exist, must stay within a finite distance from the boundary. The case $\rho^+_{\max} = \rho_{\max}$ happens only when the singularities move away from the boundary. In this scenario, the boundary seems to obstruct the existence of minimal blowup data.

4 Future research and mentoring plans

I am a principal investigator of a collaborative NSF proposal in Applied Mathematics (funding proposed for the 2022-2025 period). The project focuses on exploring the dynamics of processes of energy transfer between scales of incompressible fluid flow through investigation of natural stochastic structure that underlies deterministic Navier-Stokes equations. Similarly to the branching Brownian motion for the Fisher-KPP equation, this structure, first derived by Le Jan and Sznitman in 1997, takes the form of a branching stochastic cascade tree that governs the way information from the initial data is transferred to the NSE solution at time t. Unlike the branching Brownian motion, the NSE stochastic cascade take place in Fourier space—a natural setting to study scale-to-scale energy transfer. The theory of random trees, especially those displaying Markovian-type structures, is well-developed with a number of tools available for their study. Although exact connection between its random tree structure for wave-vector generation and the way flows modeled by the NSE solutions shed energy is not yet well-understood, the recent progress in the theory of branching cascades [6, 9–11] places such advances within reach.

The proposal contains several research problems aimed at three main directions: 1) Understanding the process of turbulent energy transfer in fluid flows through the lens of stochastic cascades; 2) Development of efficient Monte-Carlo-type simulations of solutions of Navier-Stokes (and similar) equations based on simulations of stochastic cascades; 3) Development of mean-field models to study turbulent energy transfer between scales of the fluid flows based on the probabilistic properties of branching in the stochastic cascade structure for NSE.

I will briefly explain one of our proposed research problems—the Monte Carlo method of simulating the solution to NSE. In fluid flows, the energy cascade is the transference of kinetic energy from large physical scales to small ones. Empirical theories of energy cascades based on statistical assumptions on the dynamics of fluid flows are ubiquitous in the field of hydrodynamic and aerodynamic turbulence. Although these assumptions have resulted in useful turbulence models such as the Kolmogorov 5/3 Law and the Large Eddy Simulation, their full justification at the level of the governing differential equations such as the Navier-Stokes equations remains elusive. This is partially due to the gap between the statistical nature of turbulence theories and the determinism of the fluid equations. In contrast to various existing shell models of turbulence, Le Jan-Sznitman's stochastic cascade provides a precise notion of averaging directly from NSE.

When a solution to a PDE is interpreted as the expected value of a stochastic process, Monte Carlo methods can be used as an intuitive way to simulate the solution. In the case of stochastic *branching* processes, Monte Carlo methods can be significantly slower than deterministic numerical methods such as finite difference method and finite element method. Modification techniques to obtain a practically feasible Monte Carlo algorithm include the interpolation of data and domain decomposition [1, 2, 19, 31]. A Monte Carlo implementation meets two key difficulties. First, an artificial mechanism to decide when to terminate a sampling process must be introduced. Such a stoppage mechanism not only ensures that the amount of data generated is finite but also indicates when the stochastic explosion event has occurred and the ongoing sampling process should be abandoned. Second, in order to have an adequate picture of the energy spectrum, one needs to simulate $\hat{u}(\xi, t)$ for many the wave-vectors ξ. This is where the Decoupling Principle—a property of the stochastic cascade arising from the natural scaling symmetry of NSE—plays a crucial role in reducing the cost of a Monte Carlo simulation.

There are mean-field models such as the dyadic model, α-Riccati , 1D complex Burgers,... which can be used as case study. Such a project is doable for undergraduate/graduate students. It provides an interesting applied-math framework through which students can build a strong background in differential equations, numerical simulation, and probability theory.

References

[1] Juan A. Acebrón, Ángel Rodríguez-Rozas, and Renato Spigler, *Domain decomposition solution of nonlinear two-dimensional parabolic problems by random trees*, J. Comput. Phys. **228** (2009), no. 15, 5574–5591. MR2541466

[2] ———, *Efficient parallel solution of nonlinear parabolic partial differential equations by a probabilistic domain decomposition*, J. Sci. Comput. **43** (2010), no. 2, 135–157. MR2607147

[3] David Aldous and Paul Shields, *A diffusion limit for a class of randomly-growing binary trees*, Probability Theory and Related Fields **79** (1988), no. 4, 509–542.

[4] Katharina Best and Peter Pfaffelhuber, *The Aldous-Shields model revisited with application to cellular ageing*, Electronic Communications in Probability **15** (2010), 475–488.

[5] Sergei I. Chernyshenko, Peter Constantin, James C. Robinson, and Edriss S. Titi, *A posteriori regularity of the three-dimensional Navier-Stokes equations from numerical computations*, J. Math. Phys. **48** (2007), no. 6, 065204, 15.

[6] Radu Dascaliuc, Nicholas Michalowski, Enrique Thomann, and Edward C. Waymire, *Symmetry breaking and uniqueness for the incompressible Navier-Stokes equations*, Chaos **25** (2015), no. 7, 075402, 16.

[7] ———, *Complex burgers equation: A probabilistic perspective*, Sojourns in Probability Theory and Statistical Physics - I **298** (2019), 138–170.

[8] Radu Dascaliuc, Tuan Pham, and Enrique Thomann, *On Le Jan-Sznitman's stochastic approach to the Navier-Stokes equations*, ArXiv e-prints (October 2019), available at 1910.05500.

[9] Radu Dascaliuc, Tuan N Pham, Enrique Thomann, and Edward C Waymire, *Doubly Stochastic Yule Cascades (Part I): The explosion problem in the time-reversible case* (2021), available at 2103.06912.

[10] ———, *Doubly Stochastic Yule Cascades (Part II): The explosion problem in the time-reversible case* (2021), available at 2107.13182.

[11] Radu Dascaliuc, Enrique A. Thomann, and Edward C. Waymire, *Stochastic explosion and non-uniqueness for α-Riccati equation*, J. Math. Anal. Appl. **476** (2019), no. 1, 53–85.

[12] Masoumeh Dashti and James C. Robinson, *An a posteriori condition on the numerical approximations of the Navier-Stokes equations for the existence of a strong solution*, SIAM J. Numer. Anal. **46** (2008), no. 6, 3136–3150.

[13] Hans G. Feichtinger, Karlheinz Gröchenig, Kuijie Li, and Baoxiang Wang, *Navier-Stokes equation in super-critical spaces $E^s_{p,q}$*, Ann. Inst. H. Poincaré Anal. Non Linéaire **38** (2021), no. 1, 139–173.

[14] I. Gallagher, G. Koch, and F. Planchon, *A profile decomposition approach to $L^\infty_t(L^3_x)$ Navier-Stokes regularity criterion*, Math. Ann. **355** (2013), no. 4, 1527–1559.

[15] ———, *Blow-up of critical Besov norms at a potential Navier-Stokes singularity*, Comm. Math. Phys. **343** (2016), no. 1, 39–82.

[16] H. Jia and V. Šverák, *Minimal L^3-initial data for potential Navier-Stokes singularities*, SIAM J. Math. Anal. **45** (2013), no. 3, 1448–1459.

[17] Hao Jia and Vladimír Šverák, *Local-in-space estimates near initial time for weak solutions of the Navier-Stokes equations and forward self-similar solutions*, Invent. Math. **196** (2014), no. 1, 233–265.

[18] Y. Le Jan and A. S. Sznitman, *Stochastic cascades and* 3*-dimensional Navier-Stokes equations*, Probab. Theory Related Fields **109** (1997), no. 3, 343–366.

[19] Antoine Lejay and Hernán Mardones González, *A forward-backward probabilistic algorithm for the incompressible Navier-Stokes equations*, J. Comput. Phys. **420** (2020), 109689, 19. MR4124845

[20] Buyang Li, *A bounded numerical solution with a small mesh size implies existence of a smooth solution to the Navier-Stokes equations*, Numer. Math. **147** (2021), no. 2, 283–304.

[21] Jingyue Li, Changxing Miao, and Xiaoxin Zheng, *Minimal blow-up initial data in critical Fourier-Herz spaces for potential Navier-Stokes singularities*, ArXiv e-prints (2018april), available at 1804.09842.

[22] Kuijie Li, Tohru Ozawa, and Baoxiang Wang, *Dynamical behavior for the solutions of the Navier-Stokes equation*, Commun. Pure Appl. Anal. **17** (2018), no. 4, 1511–1560.

[23] Stephen Montgomery-Smith, *Finite time blow up for a Navier-Stokes like equation*, Proc. Amer. Math. Soc. **129** (2001), no. 10, 3025–3029.

[24] Carlo Morosi and Livio Pizzocchero, *On approximate solutions of the incompressible Euler and Navier-Stokes equations*, Nonlinear Anal. **75** (2012), no. 4, 2209–2235.

[25] ———, *Smooth solutions of the Euler and Navier-Stokes equations from the a posteriori analysis of approximate solutions*, Nonlinear Anal. **113** (2015), 298–308.

[26] Tuan Pham, *Conservation of frequencies and the global regularity of the Navier-Stokes equations*. In preparation.

[27] ———, *On the explosion problem of nonhomogeneous Yule cascades*. In preparation.

[28] ———, *Topics in the regularity theory of the Navier-Stokes equations*, PhD thesis, University of Minnesota, 2018.

[29] ———, *A global regularity criterion for the Navier-Stokes equations based on approximate solutions*, ArXiv e-prints (October 2019), available at 1910.05501.

[30] Eugénie Poulon, *About the behavior of regular Navier-Stokes solutions near the blow up*, Bull. Soc. Math. France **146** (2018), no. 2, 355–390.

[31] Jorge M. Ramirez, *Multiplicative cascades applied to PDEs (two numerical examples)*, J. Comput. Phys. **214** (2006), no. 1, 122–136. MR2208673

[32] W. Rusin and V. Šverák, *Minimal initial data for potential navier-stokes singularities*, J. Funct. Anal. **260** (2011), no. 3, 879–891.

[33] G. Seregin and V. Šverák, *On global weak solutions to the Cauchy problem for the Navier-Stokes equations with large L_3-initial data*, Nonlinear Anal. **154** (2017), 269–296.

[34] G. A. Seregin, *Local regularity of suitable weak solutions to the Navier-Stokes equations near the boundary*, J. Math. Fluid Mech. **4** (2002), no. 1, 1–29.