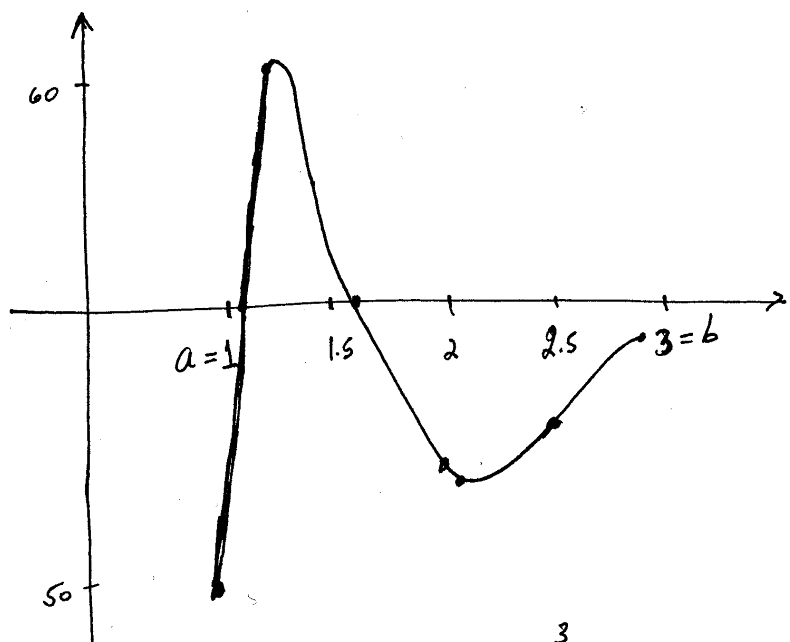
An Adaptive Simpson's scheme.

Consider a function like

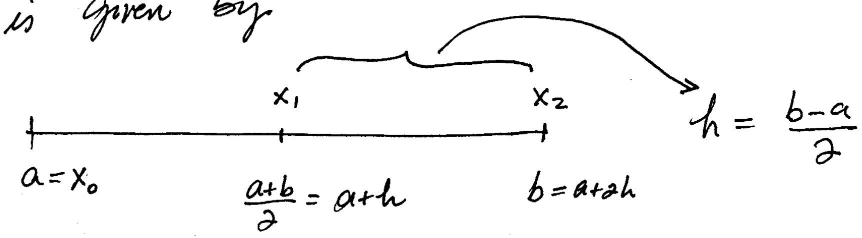
$$f(x) = \frac{100}{x^2} \sin\left(\frac{10}{x}\right)$$



We want to approximate : $\int_1^3 f(x)dx$.

Obviously ~~on~~ the subinterval $[1,2]$ the function changes more rapidly than over the subinterval $[2,3]$.
We would like to use a Simpson rule (composite) to approximate this integral. However due to the behavior of $f(x)$, we would like to use a non-uniform partition of $[1,3]$ with more points in $[1,2]$ than in $[2,3]$.

In general, Simpson over an interval $[a,b]$ for $f(x)$ is given by



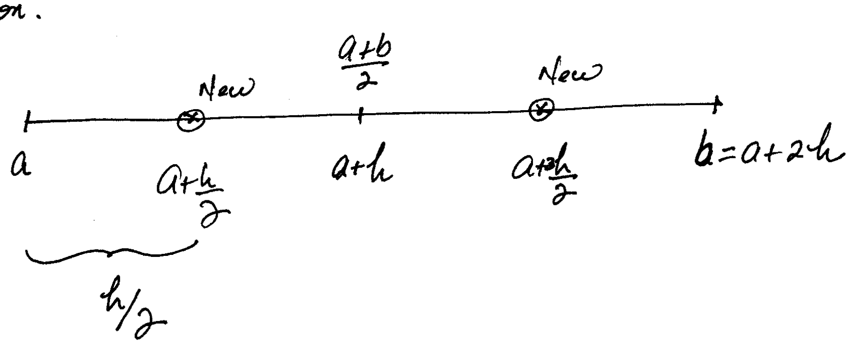
$$\int_a^b f(x)dx = \underbrace{\frac{h}{3}\left[f(a) + f(a+h) + f(b)\right]} - \frac{h^5}{90} f^{(4)}(\xi)$$

$$a < \xi < b$$

or

$$I = \int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} f^{(4)}(\xi)$$

Adding two more points and keeping a uniform partition.



We can approximate $f(x)$ with composite Simpson for $\underbrace{n=4}_{\#\text{ of subintervals.}}$ (5 points)

$$\int_a^b f(x)dx = \int_a^{a+h} f(x)\,dx + \int_{a+h}^b f(x)dx =$$

$$= \frac{h/2}{3}\left[f(a) + 4f(a+\tfrac{h}{2}) + f(a+h)\right] +$$

$$+ \frac{h/2}{3}\left[f(a+h) + 4f(a+3\tfrac{h}{2}) + f(b)\right] - \frac{b-a}{180}\left(\frac{h}{2}\right)^4 f^{(4)}(\bar{\mu}).$$

$\bar{\mu} \in (a,b)$.

$$= \frac{h}{6}\left[f(a) + 4f(a+\tfrac{h}{2}) + f(a+h)\right] +$$

$$+ \frac{h}{6}\left[f(a+h) + 4f(a+3\tfrac{h}{2}) + f(b)\right] - \underbrace{\frac{1}{90}\,\frac{b-a}{2}\,\frac{h^4}{16}\,f^{(4)}(\bar{\mu})}_{= \frac{1}{16}\frac{h^5}{90} f^{(4)}(\bar{\mu})}$$

New notation:

$$\int_a^b f(x)dx = S(a, a+h) + S(a+h, b) - \frac{1}{16}\,\frac{h^5}{90}\,f^{(4)}(\bar{\mu})$$

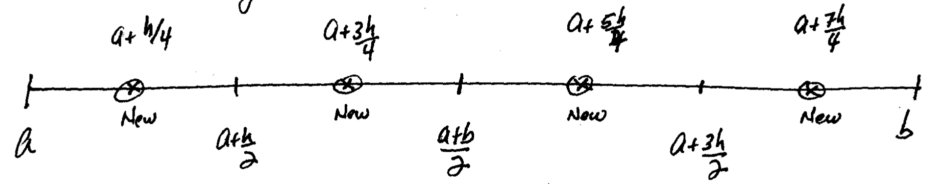
$$\therefore \quad I = S\left(a, \overset{=a+h}{\frac{a+b}{2}}\right) + S\left(\overset{=a+h}{\frac{a+b}{2}}, b\right) - \frac{1}{16}\frac{h^5}{90} f^{(4)}(\bar{\mu}) \quad (3.1)$$

Now, let's fix certain tolerance ε. We want our approximation to be within this tolerance. If somehow we could evaluate the error, then it would be easy to check if the following condition is satisfied.

$$\left| \int_a^b f(x)\,dx - S\left(a, \overset{=a+h}{\frac{a+b}{2}}\right) - S\left(\overset{=a+h}{\frac{a+b}{2}}, b\right) \right| = |\text{Error}| < \varepsilon \quad (3.2)$$

If (3.2) were not true. we subdivide the subintervals again

$h = \frac{b-a}{2}$



Now, we approximate I by Composite Simpson with $n = 8$.

$$\left| \int_a^b f(x)\,dx - S\left(a, a+\frac{h}{2}\right) - S\left(a+\frac{h}{2}, \overset{=a+h=}{\frac{a+b}{2}}\right) - S\left(\frac{a+b}{2}, a+\frac{3h}{2}\right) - S\left(a+\frac{3h}{2}, b\right) \right| \quad (3.3$$

If we substitute

$\int_a^b f(x)dx$ by $\int_a^{\frac{a+b}{2}} f(x)dx + \int_{\frac{a+b}{2}}^b f(x)dx$ in (3.3)

an applied the triangular inequality

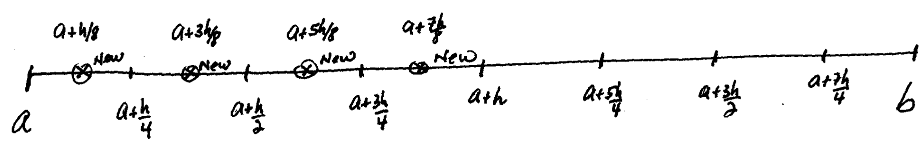
$$|(3.3)| \le \left| \int_a^{\frac{a+b}{2}} f(x)dx - S\left(a, a+\frac{h}{2}\right) - S\left(a+\frac{h}{2}, \frac{a+b}{2}\right) \right| + \longrightarrow (4.1)$$

$$+ \left| \int_{\frac{a+b}{2}}^b f(x)dx - S\left(\frac{a+b}{2}, a+\frac{3h}{2}\right) - S\left(a+\frac{3h}{2}, b\right) \right| \rightarrow (4.2)$$

Therefore, if $(4.1) < \frac{\varepsilon}{2}$ and $(4.2) < \frac{\varepsilon}{2}$.

$\Longrightarrow$ $|(3.3)| < \varepsilon$. and we are done.

But, if only one of them is less than $\varepsilon/2$, for example $(4.2) < \varepsilon/2$ but $(4.1) > \varepsilon/2$. We subdivide ~~again~~ the subinterval $[a, \frac{a+b}{2}]$ but not the subinterval $[\frac{a+b}{2}, b]$.



And then we evaluate the error

$$E = \left| \int_a^{\frac{a+b}{2}} f(x)dx - S\left(a, a+\tfrac{h}{4}\right) - S\left(a+\tfrac{h}{4}, a+\tfrac{h}{2}\right) - S\left(a+\tfrac{h}{2}, a+\tfrac{3h}{4}\right) - \right.$$
$$\left. - S\left(a+\tfrac{3h}{4}, a+h\right) \right| \quad (5.1)$$

Now, $$E \leq \left| \int_a^{a+\frac{h}{2}} f(x)dx - S\left(a, a+\tfrac{h}{4}\right) - S\left(a+\tfrac{h}{4}, a+\tfrac{h}{2}\right) \right| \rightarrow (5.2)$$
$$+ \left| \int_{a+h/2}^{a+h} f(x)dx - S\left(a+\tfrac{h}{2}, a+\tfrac{3h}{4}\right) - S\left(a+\tfrac{3h}{4}, a+h\right) \right| \rightarrow (5.3)$$

therefore, if $(5.2) < \varepsilon/4$, $(5.3) < \varepsilon/4 \Rightarrow (5.1) < \varepsilon/2$

and we are done.

$$\int_a^b f(x)dx \simeq S(a, a+h/4) + S(a+h/4, a+h/2) + S\left(a+h/2, a+\tfrac{3h}{4}\right) +$$
$$+ S\left(a+\tfrac{3h}{4}, a+h\right) + S\left(a+h, a+\tfrac{3h}{2}\right) + S\left(a+\tfrac{3h}{2}, b\right)$$

within an ε error.

Otherwise, if for example $(5.2) > \varepsilon/4$, but $(5.3) < \varepsilon/4$

We subdivide the subinterval $[a, a+\frac{h}{2}]$ but not the subinterval $[a+\frac{h}{2}, a+h]$ and repeat the previous procedure for $\varepsilon/8$.

The previous algorithm could be implemented if we could find an efficient way to obtain a bound for

$$\left| \int_a^b f(x)\,dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| \qquad (5.1.1)$$

for any interval $[a,b]$

We will prove next that

$$\boxed{\left| \int_a^b f(x)\,dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right| \simeq \frac{1}{15} \left| S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) \right|}$$

for any interval $[a,b]$. (5.1.2).

<u>Proof.</u> –

From (3.1)

$$\left|\int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| = \frac{1}{16}\frac{h^5}{90} f^{(4)}(\bar{\mu}) \quad (6.1)$$

On the other hand,

$$I = S(a,b) - \frac{h^5}{90} f^{(4)}(\xi) = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{1}{16}\frac{h^5}{90} f^{(4)}(\bar{\mu})$$

$$\Rightarrow S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right) = \frac{h^5}{90} f^{(4)}(\xi) - \frac{1}{16}\frac{h^5}{90} f^{(4)}(\bar{\mu}) \quad (6.2)$$

if $\boxed{\xi = \bar{\mu} \text{ or } f^{(4)}(\xi) \simeq f^{(4)}(\bar{\mu})}$ Key assumption.

then the right hand side in (6.2) can be simplified as

$$\frac{15}{16} f^{(4)}(\bar{\mu}) \frac{h^5}{90}$$

$$\Rightarrow \boxed{\left|\int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| = \frac{1}{15}\left|S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right|}$$

(6.3)

Therefore, if

$$\left|S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| < 15\varepsilon$$

$$\Rightarrow \left|\int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| < \varepsilon$$ and this is true for any interval [a, b].