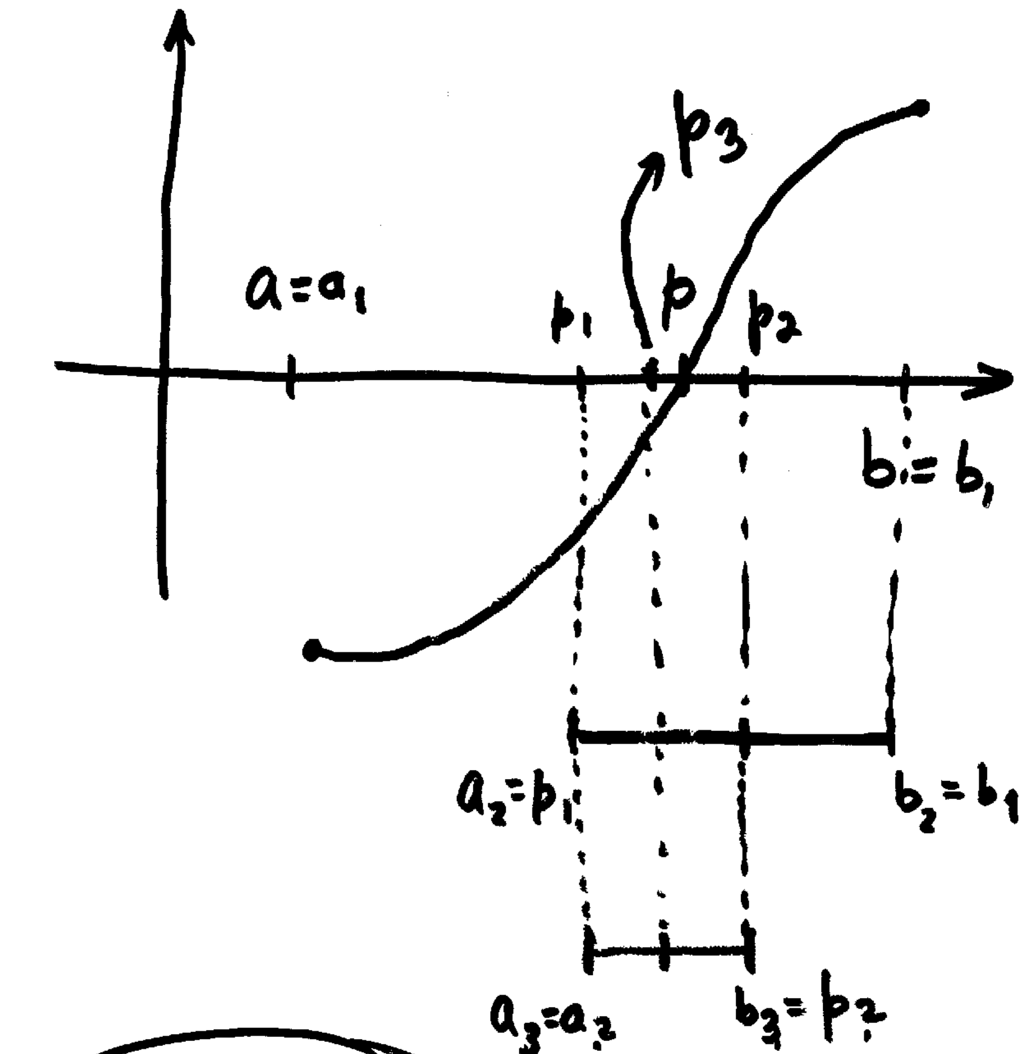


$$p_1 = \frac{a_1+b_1}{2}, \quad p_2 = \frac{a_2+b_2}{2}$$

$$\vdots$$

$$p_n = \frac{a_n+b_n}{2}$$

Cond. to check for roots based on intermediate value Thm.

Theorem.– 1) $f \in C[a,b]$

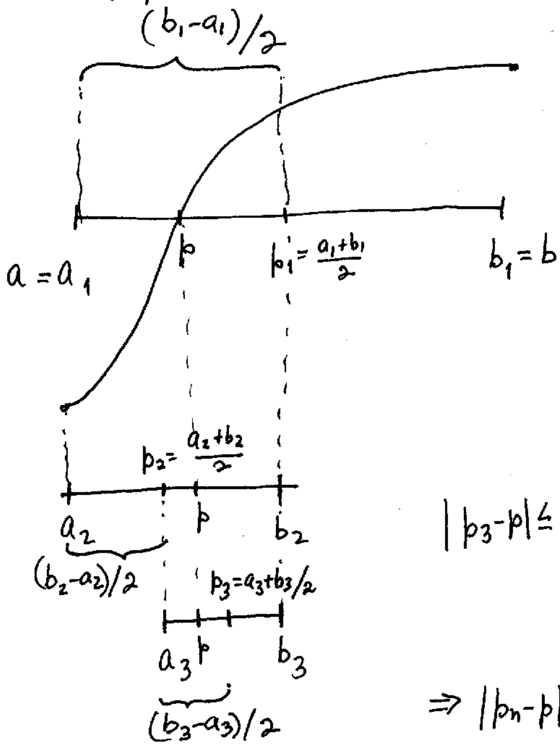
2) $f(a) * f(b) < 0$

$\Rightarrow$ there exists $p \in (a,b)$ such that $f(p) = 0$ and the bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating p with

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad n \geq 1$$

Proof.– The existence of p is guarantee by the intermediate Value theorem for Cont. funcs.

The graph below shows the convergence process



$$|p_1 - p| \leq \frac{b_1-a_1}{2}$$

$$|p_2 - p| \leq \frac{b_2-a_2}{2} = \frac{b_1-a_1}{2^2}$$

$$|p_3 - p| \leq \frac{b_3-a_3}{2} = \frac{b_2-a_2}{2^2} = \frac{b_1-a_1}{2^3}$$

$$= \frac{b-a}{2^3}$$

$$\Rightarrow |p_n - p| \leq \frac{b-a}{2^n}, \quad n \geq 1.$$

Table 2.1

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
7	1.359375	1.375	1.3671875	0.03236
8	1.359375	1.3671875	1.36328125	−0.03215
9	1.36328125	1.3671875	1.365234375	0.000072
10	1.36328125	1.365234375	1.364257813	−0.01605
11	1.364257813	1.365234375	1.364746094	−0.00799
12	1.364746094	1.365234375	1.364990235	−0.00396
13	1.364990235	1.365234375	1.365112305	−0.00194

approximation p_{13}. You might suspect this is true since $|f(p_9)| < |f(p_{13})|$, but we cannot be sure of this unless the true answer is known. ■

The Bisection method, though conceptually clear, has significant drawbacks. It is slow to converge (that is, N might become quite large before $|p - p_N|$ is sufficiently small), and a good intermediate approximation can be inadvertently discarded. However, the method has the important property that it always converges to a solution, and for that reason it is often used as a starter for the more efficient methods we will present later in this chapter.

Theorem 2.1 Suppose that $f \in C[a, b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with

$$|p_n - p| \leq \frac{b-a}{2^n}, \quad \text{when} \quad n \geq 1.$$

■