4.4 Composite Numerical Integration

COMPOSITE TRAPEZOIDAL.

Thm.– ① $f \in C^2[a,b]$

② $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ uniform $\left(= x_i - x_{i-1},\ i = 1,\dots,n \right)$

partition of $[a,b]$

Then,
$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] \underbrace{- \frac{b-a}{12} h^2 f''(\mu)}_{\text{error term}}, \quad \mu \in (a,b).$$

Proof.– For a single subinterval $[x_i, x_{i+1}]$, $i = 0,1,\dots,n-1$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx = \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right] - \frac{h^3}{12} f''(\xi_i) \qquad \xi_i \in (x_i, x_{i+1})$$

Therefore,
$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx$$

$$= \frac{h}{2} \sum_{i=0}^{n-1} \left[f(x_i) + f(x_{i+1}) \right] - \frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i) =$$

$$= \frac{h}{2} \Big[\overset{= f(a)}{f(x_0)} + f(x_1) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_{n-1}) + \overset{= f(b)}{f(x_n)} \Big] - \frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i)$$

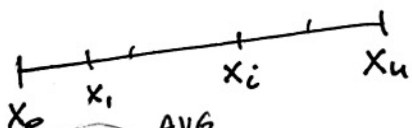
$$= \frac{h}{2} \left[f(a) + 2 \sum_{i=1}^{n-1} f(x_i) + f(b) \right] - \frac{h^3}{12} \sum_{i=0}^{n-1} f''(\xi_i)$$

Error Expression: Since $f \in C^2[a,b]$, then

$$\min_{x \in [a,b]} (f''(x)) \le f''(\xi_i) \le \max_{x \in [a,b]} (f''(x))$$

$$\Rightarrow \quad n \left(\min (f''(x)) \right) \le \sum_{i=0}^{n-1} f''(\xi_i) \le n \left(\max (f''(x)) \right)$$

Now, $h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h}$



AVG of f'' AVG

thus, $$\min (f''(x)) \le \frac{1}{n} \sum_{i=0}^{n-1} f''(\xi_i) = \frac{h}{b-a} \sum_{i=0}^{n-1} f''(\xi_i) \le \max (f''(x))$$

IVT. implies there exists $\mu \in (a,b)$ such that $$f''(\mu) = \frac{h}{b-a} \sum_{i=0}^{n-1} f''(\xi_i).$$

$$\Rightarrow \quad \sum f''(\xi_i) = \frac{b-a}{h} f''(\mu).$$

$$\Rightarrow \quad -\frac{h^3}{12}\sum_{i=0}^{n-1} f''(\xi_i) = -\frac{b-a}{\cancel{h}}\,\frac{h^{\cancel{3}\,2}}{12} f''(\mu)$$

or $\quad \text{Error} = -\frac{(b-a)}{12} h^2 f''(\mu), \quad \mu \in [a,b).$

$\hookrightarrow \underline{\underline{O(h^2)}}.$

Example 3.– (book).

$$\int_0^1 e^{-x^2} dx$$

How many points should be used to compute above integral with an error of at most $\frac{1}{2}\times 10^{-4}$?

Using the error form.

$$\left| -\frac{(b-a)}{12}\left(\frac{\overbrace{b-a}^{=1}}{n}\right)^2 f''(\mu)\right| \leq \frac{1}{2}\times 10^{-4}$$

$$\frac{1}{12}\frac{1}{n^2}|f''(\mu)| \leq \frac{1}{2}\times 10^{-4} \Rightarrow n^2 \geq \frac{2}{12}|f''(\mu)| * 10^4$$

$$f'(x) = -2x\, e^{-x^2}, \qquad f''(x) = e^{-x^2}(-2+4x^2)$$

$$\Rightarrow |f''(x)| \leq e^{-x^2}|4x^2-2| \leq 1.2 = 2$$

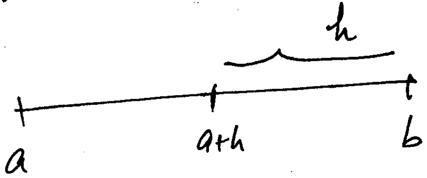
$$\therefore \quad n^2 \geq \frac{1}{3}\times 10^{4} \Rightarrow \boxed{n \geq 58}$$

Composite Simpson.

Consider first Single Simpson rule on $[a,b]$.

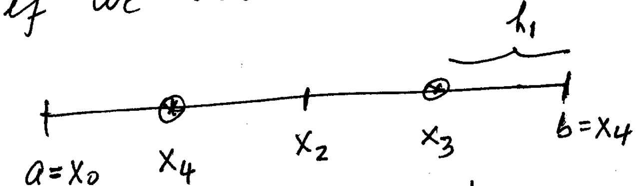
$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 4f(a+h) + f(b) \right]$$

$$h = \frac{b-a}{2}$$



Now, if we add two more points

$$h_1 = \frac{b-a}{4}$$



$$\int_a^b f(x)\,dx = \int_{x_0=a}^{x_2} f(x)\,dx + \int_{x_2}^{x_4=b} f(x)\,dx =$$

$$= \frac{h_1}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right] + \frac{h_1}{3}\left[f(x_2) + 4f(x_3) + f(x_4) \right] + E_{error}$$

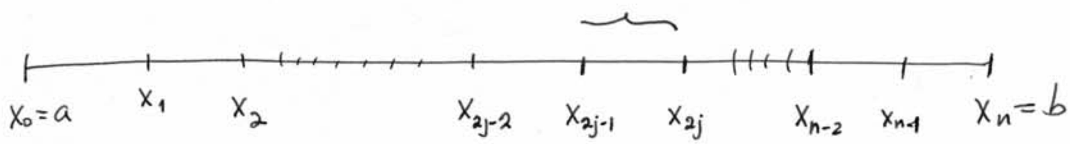
$$= \frac{h_1}{3}\left[f(x_0) + 2f(x_2) + 4\left(f(x_1) + f(x_3) \right) + f(x_4) \right] +$$

$$+ \left(-\frac{h_1}{90} f^{(4)}(\xi_1) - \frac{h_1}{90} f^{(4)}(\xi_2) \right)$$

$x_0 < \xi_1 < x_2, \quad x_2 < \xi_2 < x_4.$

In general, subdividing the interval $[a,b]$ into n "even" subintervals.

$$h = \frac{b-a}{n}$$



$$\int_a^b f(x)\,dx = \int_{x_0}^{x_2} + \int_{x_2}^{x_4} + \dots + \int_{x_{2j-2}}^{x_{2j}} + \dots + \int_{x_{n-2}}^{x_n} = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)\,dx$$

$$= \sum_{j=1}^{n/2} \left\{ \frac{h}{3}\left[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j}) \right] - \frac{h^5}{90} f^{(4)}(\xi_j) \right\}$$

$$= \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n) \right] - \frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)$$

$\xi_j \in (x_{2j-2}, x_{2j})$

It can be proved that

$$\boxed{-\frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) = -\frac{b-a}{180} h^4 f^{(4)}(\mu)}$$

follow proof for trapezoidal rule.

Summarizing,

Thm.- $f \in C^4[a,b]$, n even, $h = \frac{b-a}{n}$, $x_j = a + j*h$

$j = 0, 1, .., n.$

Then,

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right]$$

$$- \frac{b-a}{80} h^4 f^{(4)}(\mu)$$

Simpson's Rule $\mu \in (a,b)$.

Simpson's Rule:

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{(4)}(\xi).$$

Since the error term involves the fourth derivative of f, Simpson's rule gives exact results when applied to any polynomial of degree three or less.

EXAMPLE 1 The Trapezoidal rule for a function f on the interval [0, 2] is

$$\int_0^2 f(x)\,dx \approx f(0) + f(2),$$

and Simpson's rule for f on [0,2] is

$$\int_0^2 f(x)\,dx \approx \frac{1}{3}[f(0) + 4f(1) + f(2)].$$

The results to three places for some elementary functions are summarized in Table 4.7. Notice that in each instance Simpson's Rule is significantly better. ■

Table 4.7

$f(x)$	x^2	x^4	$1/(x+1)$	$\sqrt{1+x^2}$	$\sin x$	e^x
Exact value	2.667	6.400	1.099	2.958	1.416	6.389
Trapezoidal	4.000	16.000	1.333	3.326	0.909	8.389
Simpson's	2.667	6.667	1.111	2.964	1.425	6.421