Eingenvalues and Eigenvectors.

Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

If we multiply A by $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

we obtain

$$A\vec{v} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad A\vec{y} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

So $\quad A\vec{v} = \vec{v} \quad , \quad A\vec{y} = \vec{y}$

However, $A \neq I$. If we pick another vector, for example $\vec{z} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, then

$$A\vec{z} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

and for $\vec{\omega} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A\vec{\omega} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Also, for $\vec{x} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$$A\vec{x} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

Therefore, $A\vec{\omega} = 4\vec{\omega}$ and $A\vec{x} = 4\vec{x}$.

Obviously, the vectors $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are special vectors. But not only these two vectors have special properties when A is multiplied by them. <u>Every multiple of them has the same property</u>. In fact

$$A(\alpha\vec{v}) \overset{\text{matrix arithmetic}}{=} \alpha A\vec{v} = \alpha\vec{v}, \quad \text{for any } \alpha \text{ real.}$$

$$A(\alpha\vec{\omega}) = \alpha A\vec{\omega} = \alpha 4\vec{\omega} = 4(\alpha\vec{\omega}), \quad \text{for any } \alpha$$

Vectors as $\vec{v}$ and $\vec{w}$ receive the special name of eigenvectors of the matrix A. Also, the scalar numbers $\lambda_1 = 1$, $\lambda_2 = 4$ receive the special name of eigenvalues of the matrix A.

Question: Given a matrix $A_{(n \times n)}$, How can we find its eigenvalues and eigenvectors?

We are looking for scalars λ and vectors $\vec{v}$ that satisfy the equation

$$\boxed{A\vec{x} = \lambda\vec{x}} \qquad (9.1)$$

This equation is equivalent to

$$A\vec{x} - \lambda\vec{x} = 0 \quad \text{or} \quad A\vec{x} - \lambda I\vec{x} = 0$$

$$\text{or} \quad \boxed{(A - \lambda I)\vec{x} = \vec{0}} \quad \text{or} \quad \boxed{B\vec{x} = \vec{0}} \qquad (9.2)$$

where $B = A - \lambda I$

(9.2) is a homogeneous linear system of equations

Obviously, $\vec{x} = \vec{0}$ is a solution of (9.1) for any λ

We are interested in finding non-trivial Solutions of

$$A\hat{x} = \lambda \hat{x} \qquad (9.1)$$

or

$$\left.\begin{aligned}(A-\lambda I)\,\vec{x} &= \vec{0} \\ B\,\hat{x} &= \vec{0}\end{aligned}\right\} \; (9.2) \quad B = A-\lambda I$$

What condition should we impose on B to guarantee the existence of non-trivial solutions?

$$\det(B) = \det(A-\lambda I) = |A-\lambda I| = 0. \quad (10.1)$$

This condition reduces to an equation for λ. By finding the Solutions of (10.1), the eigenvalues are determined. Once an eigenvalue has been determined. It can be substituted in (9.2) to find its non-trivial Solutions or eigenvectors.

Back to our example to illustrate the procedure.

If $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$, then

$$|A - \lambda I| = \left| \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda) - 2 =$$
$$= (\lambda-3)(\lambda-2) - 2 = \lambda^2 - 5\lambda + 6 - 2$$
$$= \lambda^2 - 5\lambda + 4 = (\lambda-1)(\lambda-4)$$

Therefore, $|A - \lambda I| = 0$ if $\lambda_1 = 1$, $\lambda_2 = 4$.

Now, we will find the eigenvectors associated to each one of the eigenvalues.

For $\lambda_1 = 1$, we have to solve the homogeneous system $(A - 1.I)\vec{x} = \vec{0}$ (11.1)

We know, this system has non-trivial solutions Why?

or

$$\begin{bmatrix} 3-1 & 1 \\ 2 & 2-1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{cases} 2V_1 + V_2 = 0 \\ 2V_1 + V_2 = 0 \end{cases}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} 2V_1 + V_2 = 0 \\ V_2 = -2V_1 \end{array}$$

or $\vec{V} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} t \\ -2t \end{pmatrix}$

for any $t \neq 0$.

This is a family of eigenvectors associated to the eigenvalue $\lambda_1 = 1$.

We will do the same work to find the eigenvectors for $\lambda_2 = 4$

$$\begin{bmatrix} 3-4 & 1 \\ 2 & 2-4 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

row-reducing

$$\begin{cases} -w_1 + w_2 = 0 \\ 2w_1 - 2w_2 = 0 \end{cases}$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 2 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow -w_1 + w_2 = 0 \Rightarrow w_2 = w_1$ or $\vec{w} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, for any $t \neq 0$.

or $\vec{w} = \begin{pmatrix} t \\ t \end{pmatrix}$