4.7 Gaussian Quadrature

In general, all quad. rules are a particular case of

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i) \qquad (1)$$

Important question: Best choices of c_i's and x_i's for higher degree of precision.

For example, for $n=2$: $\underline{x_1, x_2}$ and $[a,b]=[-1,1]$, what should be x_1, x_2, c_1, c_2 such that

$$\int_{-1}^{1} f(x)dx \simeq c_1 f(x_1) + c_2 f(x_2) \qquad (2)$$

is of highest degree of precision?

Since we want to determine 4 unknowns, it might be that degree of precision for an optimal choice is 3. It means we will try to determine x_1, x_2, c_1, c_2 such that (2) is exact for $1, x, x^2$, and x^3.

We arrive to the following system of equations:

$$\begin{cases} C_1 + C_2 = \int_{-1}^{1} dx = 2 \\ C_1 x_1 + C_2 x_2 = \int_{-1}^{1} x\,dx = 0 \\ C_1 x_1^2 + C_2 x_2^2 = \int_{-1}^{1} x^2 dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3} \\ C_1 x_1^3 + C_2 x_2^3 = \int_{-1}^{1} x^3 dx = 0 \end{cases}$$

Nonlinear system. Solutions are: $C_1 = 1$, $x_1 = -\frac{\sqrt{3}}{3}$, $C_2 = 1$, $x_2 = \frac{\sqrt{3}}{3}$.

Therefore, quadrature rule using only two nodes with highest degree of precision is

$$\boxed{\int_{-1}^{1} f(x)\,dx \simeq f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).} \qquad (3)$$

We can generate higher degree of precision formulas by following a similar procedure. But, there is an EASIER WAY.

- Notice that $x_1 = -\frac{\sqrt{3}}{3}$ and $x_2 = \frac{\sqrt{3}}{3}$ are the two roots of $P_2(x) = x^2 - \frac{1}{3}$ (Legendre polyn. degree 2). They gave quad. formula exact to polynomials of degree less than 4.

- Show other Legendre polynomials (MAPLE).

In general, a quadrature formula exact for polynomials of degree less than $2n$ can be obtained using the roots: $x_1, .., x_n$ of Legendre polynomials of degree n. How about the coefficients C_i's ?

Thm 4.7

1) $x_1, x_2, , x_n$ are roots of Legendre polynomial $P_n(x)$

2) The Coefficients C_i, $i=1,,n$ are defined by

$$C_i = \int_{-1}^{1} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(x-x_j)}{x_i - x_j} \right) dx$$

Then,

if $P(x)$ is any polynomial of degree less than $2n$

$$\int_{-1}^{1} P(x)\,dx \underset{\text{"exactly"}}{=} \sum_{i=1}^{n} C_i P(x_i)$$

It means that the formula

$$\int_{-1}^{1} f(x)\,dx \simeq \sum_{i=1}^{n} C_i f(x_i)$$

is at least of degree of precision $2n-1$.

Therefore, (3) is a Quad. Gauss formula with two nodes ($n=2$) and degree of precision 3 ($2n-1$).

The next formula, by using the previous thm is given by

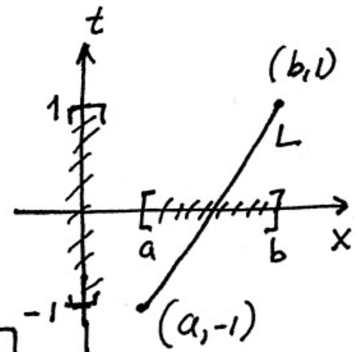
$$\int_{-1}^{1} f(x)dx \simeq \frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right)$$

$$\simeq 0.556\, f(-0.77) + 0.889\, f(0) + 0.556\, f(0.77)$$

with degree of precision $2(3)-1 = 5$.

Gaussian Quadrature on any Interval

Approx $\int_a^b f(x)dx$ using Gauss quad.



I) Transform $[a,b] \longrightarrow [-1,1]$

$x \longrightarrow t(x)$

Slope of L: $m = \frac{2}{b-a}$

L: $t-1 = \frac{2}{b-a}(x-b) \Rightarrow x-b = \left(\frac{b-a}{2}\right)\cdot(t-1)$

$$\boxed{t = \frac{2}{b-a}(x-b)+1}$$

or $\boxed{x = \frac{b-a}{2}(t-1) + b}$, $t \in [-1,1]$

$\Rightarrow \quad dx = \frac{b-a}{2}dt$

Thus, changing variables $x \to t$

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2}(t-1)+b\right)\left(\frac{b-a}{2}\right)dt$$

In particular, for

$$\int_{1=a}^{1.5=b} e^{-x^2}\,dx$$

$$\frac{b-a}{2} = \frac{1}{4}$$

$$x=1 \to t=-1$$
$$x=1.5 \to t=1$$

$$x = \frac{b-a}{2}(t-1)+b = \frac{1}{4}(t-1)+\frac{3}{2} = \frac{t-1+6}{4} = \frac{t+5}{4}$$

Thus,

$$\int_1^{1.5} e^{-x^2}\,dx = \int_{-1}^{1} e^{-\left(\frac{t+5}{4}\right)^2}\frac{1}{4}\,dt \simeq$$

Gauss 2 points

$$\simeq e^{-\frac{\left(-\frac{\sqrt{3}}{3}+5\right)^2}{16}} + e^{-\frac{\left(\frac{\sqrt{3}}{3}+5\right)^2}{16}} \simeq 0.1094003$$

Gauss 3 points

$$\simeq \frac{5}{9}e^{-\frac{\left(-\sqrt{\frac{3}{5}}+5\right)^2}{16}} + \frac{8}{9}e^{-\frac{(0+5)^2}{16}} +$$

$$+ \frac{5}{9}e^{-\frac{\left(\sqrt{\frac{3}{5}}+5\right)^2}{16}}$$

$$\simeq 0.1093642$$

Table 4.11

n	Roots $r_{n,i}$	Coefficients $c_{n,i}$
2	0.5773502692	1.0000000000
	−0.5773502692	1.0000000000
3	0.7745966692	0.5555555556
	0.0000000000	0.8888888889
	−0.7745966692	0.5555555556
4	0.8611363116	0.3478548451
	0.3399810436	0.6521451549
	−0.3399810436	0.6521451549
	−0.8611363116	0.3478548451
5	0.9061798459	0.2369268850
	0.5384693101	0.4786286705
	0.0000000000	0.5688888889
	−0.5384693101	0.4786286705
	−0.9061798459	0.2369268850

Example 1 Approximate $\int_{-1}^{1} e^x \cos x\,dx$ using Gaussian quadrature with $n = 3$. The entries in Table 4.11 give us

$$\int_{-1}^{1} e^x \cos x\,dx \approx 0.\overline{5}e^{0.774596692}\cos 0.774596692$$

$$+\,0.\overline{8}\cos 0 + 0.\overline{5}e^{-0.774596692}\cos(-0.774596692)$$

Table 4.12

n	0	1	2	3	4
Closed formulas		0.1183197	0.1093104	0.1093404	0.1093643
Open formulas	0.1048057	0.1063473	0.1094116	0.1093971	

The Gaussian quadrature procedure applied to this problem requires that the integral first be transformed into a problem whose interval of integration is $[-1, 1]$. Using Eq. (4.42) gives

$$\int_1^{1.5} e^{-x^2}\, dx = \frac{1}{4}\int_{-1}^{1} e^{(-(t+5)^2/16)}\, dt.$$

The values in Table 4.11 give the following Gaussian quadrature approximations for this problem:

***n* = 2:**

$$\int_1^{1.5} e^{-x^2}\, dx \approx \frac{1}{4}[e^{-(5+0.5773502692)^2/16} + e^{-(5-0.5773502692)^2/16}] = 0.1094003;$$

***n* = 3:**

$$\int_1^{1.5} e^{-x^2}\, dx \approx \frac{1}{4}[(0.5555555556)e^{-(5+0.7745966692)^2/16} + (0.8888888889)e^{-(5)^2/16}$$

$$+ (0.5555555556)e^{-(5-0.7745966692)^2/16}]$$

$$= 0.1093642.$$