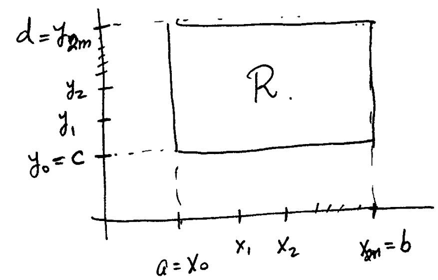
Composite Simpson's Rule in 2-D.



$h = \frac{b-a}{2n}$, $K = \frac{d-c}{2m}$

$$\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx =$$

$$= \frac{K}{3} \left[\int_a^b f(x, y_0)\, dx + 2 \sum_{j=1}^{m-1} \int_a^b f(x, y_{2j})\, dx + \right.$$

$$\left. + 4 \sum_{j=1}^{m} \int_a^b f(x, y_{2j-1})\, dx + \int_a^b f(x, y_{2m})\, dx \right]$$

$$- \frac{(d-c) K^4}{180} \int_a^b \frac{\partial^4 f(x, \mu)}{\partial y^4}\, dx \qquad (1)$$

Composite Simpson's rule in 2D.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \frac{K}{3}\left[I_0 + I_{2j} + I_{2j-1} + I_{2m}\right] - \frac{(b-a)(d-c)}{180}\left[h^4 \frac{\partial^4 f}{\partial x^4}(\hat{\eta},\hat{\mu}) + K^4 \frac{\partial^4 f}{\partial y^4}(\eta,\mu)\right]$$

$a < \hat{\eta}, \eta < b$

$c < \mu, \hat{\mu} < d$

where

$$I_0 = \frac{h}{3}\left[f(x_0,y_0) + 2\sum_{i=1}^{n-1} f(x_{2i},y_0) + 4\sum_{i=1}^{n} f(x_{2i-1},y_0) + f(x_{2n},y_0)\right]$$

$$I_{2j} = \frac{2h}{3}\left[\sum_{j=1}^{m-1} f(x_0,y_{2j}) + 2\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f(x_{2i},y_{2j}) + 4\sum_{j=1}^{m-1}\sum_{i=1}^{n} f(x_{2i},y_{2j}) + \sum_{j=1}^{m-1} f(x_{2n},y_{2j})\right]$$

$$I_{2j-1} = \frac{4h}{3}\left[\sum_{j=1}^{m} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{m}\sum_{i=1}^{n-1} f(x_{2i},y_{2j-1}) + 4\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{2i-1},y_{2j-1}) + \sum_{j=1}^{m} f(x_{2n},y_{2j-1})\right]$$

$$I_{2m} = \frac{h}{3}\left[f(x_0,y_{2m}) + 2\sum_{i=1}^{n-1} f(x_{2i},y_{2m}) + 4\sum_{i=1}^{n} f(x_{2i-1},y_{2m}) + f(x_{2n},y_{2m})\right]$$

Each Single Term transforms after integration:

(I)
$$\int_a^b f(x,y_0)\,dx = \frac{h}{3}\left\{f(x_0,y_0) + 2\sum_{i=1}^{m-1} f(x_{2i},y_0) + 4\sum_{i=1}^{n} f(x_{2i-1},y_0) + f(x_{2n},y_0)\right\} +$$
$$- \frac{b-a}{180} h^4 \frac{\partial^4 f}{\partial x^4}(\xi_0, y_0) = I_0 + E_0$$

(II)
$$2\sum_{j=1}^{m-1}\int_a^b f(x,y_{2j})\,dx = \frac{2h}{3}\left\{\sum_{j=1}^{m-1} f(x_0,y_{2j}) + 2\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f(x_{2i},y_{2j}) + 4\sum_{j=1}^{m-1}\sum_{i=1}^{n} f(x_{2i-1},y_{2j}) +\right.$$
$$\left. + \sum_{j=1}^{m-1} f(x_{2n},y_{2j})\right\} + 2\sum_{j=1}^{m-1}(-1)\frac{b-a}{180} h^4 \frac{\partial^4 f}{\partial x^4}(\xi_{2j}, y_{2j}) = I_{2j} + E_{2j}$$

(III)
$$4\sum_{j=1}^{m}\int_a^b f(x,y_{2j-1})\,dx = \frac{4h}{3}\left\{\sum_{j=1}^{m} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{m}\sum_{i=1}^{n-1} f(x_{2i},y_{2j-1}) + 4\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{2i-1},y_{2j-1}) +\right.$$
$$\left. + \sum_{j=1}^{m} f(x_{2n},y_{2j-1})\right\} + 4\sum_{j=1}^{m}(-1)\frac{b-a}{180} h^4 \frac{\partial^4 f}{\partial x^4}(\xi_{2j-1}, y_{2j-1}) =$$
$$= I_{2j-1} + E_{2j-1}$$

IV

$$\int_a^b f(x, y_{2m})\,dx = \frac{h}{3}\left\{ f(x_0, y_{2m}) + 2\sum_{i=1}^{n-1} f(x_{2i}, y_{2m}) + 4\sum_{i=1}^{n} f(x_{2i-1}, y_{2m}) + \right.$$

$$\left. + f(x_{2n}, y_{2m}) \right\} - \frac{b-a}{180} h^4 \frac{\partial^4 f}{\partial x^4}(\xi_{2m}, y_{2m}) = I_{2m} + E_{2m}$$

Substitution on (1) leads to

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \frac{k}{3}\left[I_0 + I_{2j} + I_{2j-1} + I_{2m} \right] + \frac{k}{3}\left[E_0 + E_{2j} + E_{2j-1} + E_{2m} \right]$$

$$\underbrace{- \frac{(d-c)k^4}{180} \int_a^b \frac{\partial^4 f}{\partial y^4}(x, \mu)\,dx}_{= E_x} \qquad (2)$$

Assuming $\frac{\partial^4 f}{\partial x^4}(x,y)$ is Cont. on R.

there exist $\min\limits_{(x,y)\in R} \frac{\partial^4 f}{\partial x^4}(x,y)$ and $\max\limits_{(x,y)\in R} \frac{\partial^4 f}{\partial x}(x,y)$

On the other hand,

$$\frac{K}{3}\left[E_0 + E_{2j} + E_{2j-1} + E_{2m}\right] = \frac{-K}{3}\,\frac{(b-a)}{180}\,h^4\left[\frac{\partial^4 f}{\partial x^4}(\xi_0, y_0) + \right.$$

$$\left. + 2\sum_{j=1}^{m-1}\frac{\partial^4 f}{\partial x^4}(\xi_{2j}, y_{2j-1}) + 4\sum_{j=1}^{m}\frac{\partial^4 f}{\partial x^4}(\xi_{2j-1}, y_{2j-1}) + \frac{\partial^4 f}{\partial x^4}(\xi_{2m}, y_{2m})\right]$$

$$= -\frac{K}{3}\,\frac{(b-a)}{180}\,h^4\left[I_{4,0} + 2\,I_{4,2j} + 4\,I_{4,2j-1} + I_{4,2m}\right] = \frac{-K}{3}\,\frac{(b-a)}{180}\,h^4\,\tilde{I}_4$$

Now,

$$(1+2(m-1)+4m+1)\min_R \frac{\partial^4 f}{\partial x^4} \le \tilde{I}_4 \le (1+2(m-1)+4m+1)\max_R \frac{\partial^4 f}{\partial x^4}$$

then

$$\min_R \frac{\partial^4 f}{\partial x^4} \le \frac{1}{6m}\tilde{I}_4 \le \max_R \frac{\partial^4 f}{\partial x^4}$$

$\therefore$

Intermediate Value thm $\Rightarrow$ $\frac{1}{6m}\tilde{I}_4 = \frac{\partial^4 f}{\partial x^4}(\hat{\eta}, \hat{\mu})$, $\quad(\hat{\eta}, \hat{\mu})$ in R.

Since, $k = \frac{d-c}{2m} \Rightarrow 6m = 3\,\frac{d-c}{k}$

$$\therefore \quad \frac{k}{3}\left[E_0 + E_{2j} + E_{2j-1} + E_{2m}\right] = -\frac{\cancel{k}}{\cancel{3}}\,\frac{(b-a)}{180}\,h^4\,\frac{\cancel{3}(d-c)}{\cancel{k}}\,\frac{\partial^4 f}{\partial x^4}(\hat{\eta},\hat{\mu})$$

$$= -\frac{(b-a)(d-c)}{180}\,h^4\,\frac{\partial^4 f}{\partial x^4}(\hat{\eta},\hat{\mu}).$$

Also, by using the Mean Value thm for integrals

$$-\frac{(d-c)k^4}{180}\int_a^b \frac{\partial^4 f}{\partial y^4}(x,\mu)\,dx = -\frac{(d-c)(b-a)}{180}\,k^4\,\frac{\partial^4 f}{\partial y^4}(\eta,\mu)$$

Finally,

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \frac{k}{3}\left[I_0 + I_{2j} + I_{2j-1} + I_{2m}\right] - \frac{(b-a)(d-c)}{180}\left[h^4\frac{\partial^4 f}{\partial x^4}(\hat{\eta},\hat{\mu}) + k^4\frac{\partial^4 f}{\partial y^4}(\eta,\mu)\right]$$

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where

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$$I_{2j} = \frac{2h}{3}\left[\sum_{j=1}^{m-1} f(x_0,y_{2j}) + 2\sum_{j=1}^{m-1}\sum_{i=1}^{n-1} f(x_{2i},y_{2j}) + 4\sum_{j=1}^{m-1}\sum_{i=1}^{n} f(x_{2i},y_{2j}) + \sum_{j=1}^{m-1} f(x_{2n},y_{2j})\right]$$

$$I_{2j-1} = \frac{4h}{3}\left[\sum_{j=1}^{m} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{m}\sum_{i=1}^{n-1} f(x_{2i},y_{2j-1}) + 4\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_{2i-1},y_{2j-1}) + \sum_{j=1}^{m} f(x_{2n},y_{2j-1})\right]$$

$$I_{2m} = \frac{h}{3}\left[f(x_0,y_{2m}) + 2\sum_{i=1}^{n-1} f(x_{2i},y_{2m}) + 4\sum_{i=1}^{n} f(x_{2i-1},y_{2m}) + f(x_{2n},y_{2m})\right]$$