Newton - Raphson Method

(Exercise 11 Sect. 2.3)

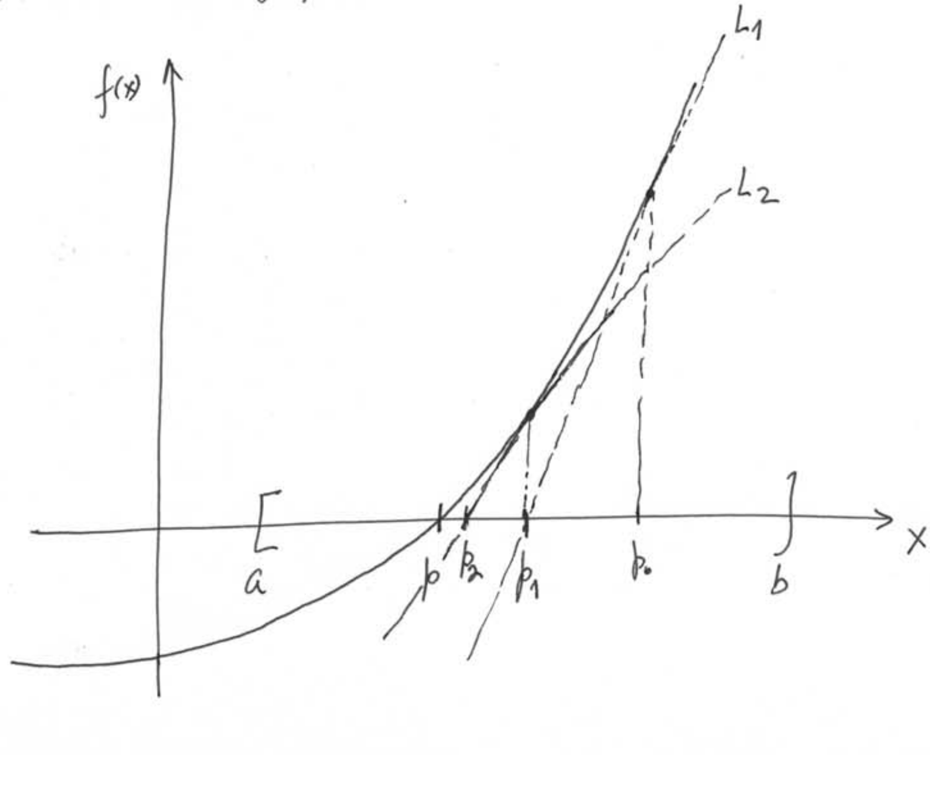
Assumptions: Given $y = f(x) \in C^2[a,b]$

f, f' and f'' are continuous in $[a,b]$.

and knowing $f(x) = 0$ has a root $p \in [a,b]$

and $f'(x) \neq 0$ on (a,b).

Then, we can construct a sequence of tangent lines to the graph of $f(x)$ as follows:



<u>Steps:</u> $f(a) * f(b) < 0$

1) Choose $p_0 \in (a,b)$

2) Construct tangent line through $(p_0, f(p_0))$: L_1

$$y - f(p_0) = f'(p_0)(x - p_0)$$

or $$y = f(p_0) + f'(p_0)(x - p_0).$$

3) Find intersection of L_1 with x-axis.

$\underline{\underline{y=0}}$ $$(x^* - p_0) f'(p_0) = -f(p_0)$$

$$\Rightarrow x^* = p_0 - \frac{f(p_0)}{f'(p_0)} = p_1$$

4) Call $x^* = p_1$ and repeat process to construct L_2 through $(p_1, f(p_1))$ and find p_2 as the intersection of L_2 with the x-axis., i.e,

$$\boxed{p_2 = p_1 - \frac{f(p_1)}{f'(p_1)}}$$

After n steps $$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}}$$ functional iteration.

Newton Iteration

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_n)} \qquad (N.1)$$

It can be seen as a fixed point problem

$$g(x) = x - \frac{f(x)}{f'(x)} \qquad (N.2)$$

assuming $f'(x) \neq 0$.

It's clear from (N.2) the equivalence between the problems of finding a root p of $f(x)$ and finding a fixed point p of $g(x)$.

<u>theorem.</u>– (Newton Method)

1. $f \in C^2[a,b]$ (Twice continuosly differentiable)
2. there exists p in (a,b), such that $f(p) = 0$.
3. $f'(p) \neq 0$.

Then, there exists $\delta > 0$, such that for $p_0 \in [p-\delta, p+\delta]$ Newton iteration (N.1) generates $\{p_n\}_{n=1}^{\infty}$ such that

$$p_n \xrightarrow[n \to \infty]{} p$$

Theorem :- (Fixed point)

① $g(x)$ is a Continuous function on $[a,b]$.

② The Values $g(x)$ belong to $[a,b]$. $g(x) \in [a,b]$

③ $g'(x)$ exists in (a,b)

④ there exists $0 < K < 1$, Such that

$$|g'(x)| \leq K, \quad x \in (a,b)$$

then

① there is a unique fixed point $p \in [a,b]$.

② the Sequence $\{p_n\}_{n=1}^{\infty}$ obtained from the functional iteration

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

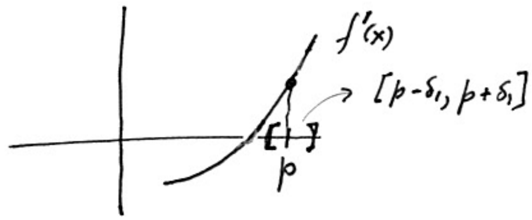
where p_0 an arbitrary point in $[a,b]$, Converges to the unique point p in $[a,b]$.

Proof.—

Since $f'(p) \neq 0$ and $f'(x)$ is cont in $[a, b]$

then, there exists $\delta_1 > 0$ such that $f'(x) \neq 0$

in $[p - \delta_1, p + \delta_1]$



Therefore,

$$g(x) = x - \frac{f(x)}{f'(x)}$$

is cont in $[p - \delta_1, p + \delta_1]$

because x, $f(x)$ and $f'(x)$ are all conts. in $[p - \delta_1, p + \delta_1]$

and $f'(x) \neq 0$ in $[p - \delta_1, p + \delta_1]$.

Next, we will find an interval $[p - \delta, p + \delta] \subset [p - \delta_1, p + \delta_1]$

such that

a) $g(x) \in C[p - \delta, p + \delta]$, b) $g([p - \delta, p + \delta]) \subset [p - \delta, p + \delta]$

c) $|g'(x)| \leq K < 1$

Once this conditions have been established, application of the fixed point theorem to $g(x)$ leads to the conclusion that the functional iteration of $g(x)$

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \xrightarrow[n \to \infty]{} p.$$

Condition (a)

To establish condition (a) is trivial, since

if $g(x) \in C[p-\delta_1, p+\delta_1]$, then $g(x) \in C[p-\delta, p+\delta]$

for any $\delta > 0$ such that $[p-\delta, p+\delta] \subset [p-\delta_1, p+\delta_1]$.

Condition (c)

Applying quotient rule to $g(x)$

$$g'(x) = 1 - \frac{[f'(x)]^2 - f(x)\, f''(x)}{[f'(x)]^2} = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

This formula for $g'(x)$ is valid in $[p-\delta_1, p+\delta_1]$

because $f''(x)$ exists there and $f'(x) \neq 0$.

Thus $g'(x)$ exists in $[p-\delta_1, p+\delta_1]$.

Moreover, $\boxed{g'(x) \text{ is conts. in } [p-\delta_1, p+\delta_1]}$ because $\rightarrow$ (*)

f, f', and f'' are conts and $f'(x) \neq 0$ in $[p-\delta_1, p+\delta_1]$.

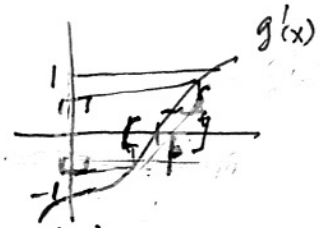
Furthermore,

$$\boxed{g'(p) = \frac{f(p)\, f''(p)}{[f'(p)]^2} = 0,}$$

(by Hipothesis $f'(p) \neq 0$)

(**)

From (*) and (**), we conclude that there exists an interval $[p-\delta, p+\delta] \subset [p-\delta_1, p+\delta_1]$



where $\boxed{|g'(x)| \leq K < 1}$ Condition (c)

Condition (b)

Finally, using that $g(x) \in C[p-\delta, p+\delta]$ (cond. (a)) and that $g'(x)$ exists in $(p-\delta, p+\delta)$

For any $x \in [p-\delta, p+\delta]$:

$$|g(x)-p| = |g(x)-g(p)| \overset{MVT}{=} g'(\xi)\,|x-p| \leq K\,|x-p|$$
$$\xi \in (p-\delta, p+\delta)$$

and $\overset{<1}{K}|x-p| < |x-p| < \delta$

Therefore $|g(x)-p| < \delta$, which is equivalent to

$$g(x) \in (p-\delta, p+\delta) \subset [p-\delta, p+\delta] \quad \text{(Condition (b))}$$

Since all hypothesis of fixed point theorem (a), (b), and (c) are satisfied, then the functional iteration

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \xrightarrow[n\to\infty]{} p.$$

Orders of Convergence.

Assume $\{x_n\}_{n=1}^{\infty}$ is converging to x^*.

(I) $\{x_n\}$ converges to x^* at least linear if there are $0<c<1$, $N>0$ such that

$$|x_{n+1}-x^*| \le c\,|x_n-x^*|, \quad n \ge N.$$

(II) $\{x_n\}$ Converges to x^* at least Superlinear if there exist a sequence $\varepsilon_n \to 0$ and $N>0$ Such that

$$|x_{n+1}-x^*| \le \varepsilon_n\,|x_n-x^*|$$

(III) $\{x_n\}$ Converges to x^* at least quadratic if there are a constant C (not necessarily less than 1) and an integer N such that

$$|x_{n+1}-x^*| \le C\,|x_n-x^*|^2$$

Convergence of Newton Method. assuming $f \in C^3[a,b]$.

For $x \in [p-\delta, p+\delta]$

$$g(x) = g(p) + (x-p)\,g'(p) + \frac{(x-p)^2}{2}\,g''(\xi(x))$$

$\xi(x)$ btw x and p

1^{st} order Taylor polynomial + reminder.

Since $g'(p)=0$ and $g(p)=p$.

$$g(x) = p + \frac{(x-p)^2}{2}\,g''(\xi(x))$$

For $x = p_n$

$$p_{n+1} = g(p_n) = p + \frac{(p_n-p)^2}{2}\,g''(\xi_n),$$

ξ_n btw p and p_n.

$$\Rightarrow \left|p_{n+1}-p\right| = \frac{1}{2}\left|p_n-p\right|^2 \left|g''(\xi_n)\right|$$

$$\Rightarrow \frac{\left|p_{n+1}-p\right|}{\left|p_n-p\right|^2} = \frac{1}{2}\left|g''(\xi_n)\right|$$

$$\Rightarrow \lim_{n\to\infty} \frac{\left|p_{n+1}-p\right|}{\left|p_n-p\right|^2} \overset{\text{Cont. of } g''(x)}{=} \frac{1}{2}\left|g''(p)\right| \Rightarrow \boxed{p_n \to p \text{ Quadratically.}}$$

A more general proof, requiring only $f \in C^2[a,b]$ is in Cheney-Kincaid pp. 107-108.