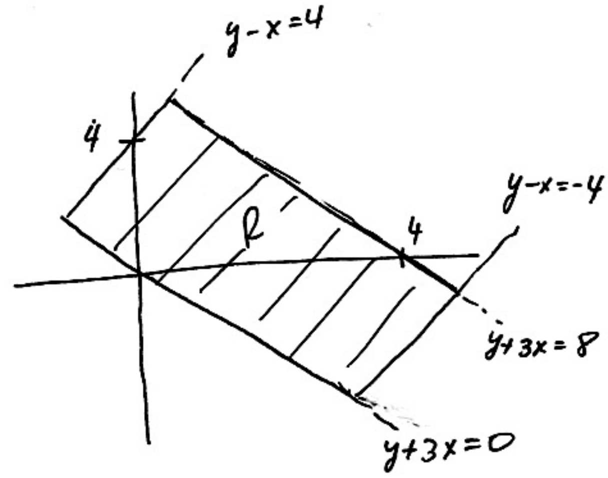
15.9 Change of Variables for Multiple Integrals.

Summary

Want to compute

$$I = \iint_R (4x+8y)\, dA, \quad \text{where}$$

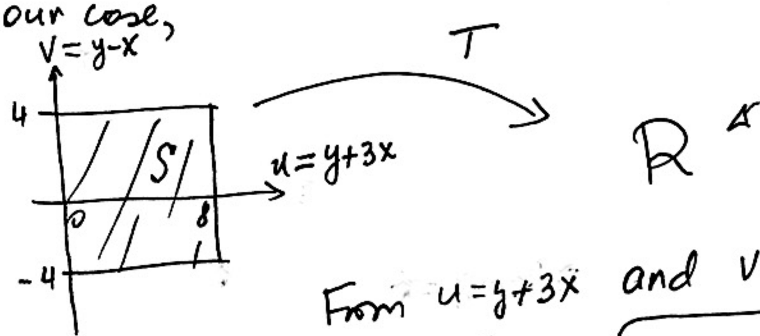


Step 1.- Find an appropriate transformation

$$T: S \longrightarrow R$$

$$(u,v) \longrightarrow (x,y) \quad x = x(u,v), \quad y = y(u,v).$$

In our case,



From $u = y+3x$ and $v = y-x$

obtain $\boxed{x = \frac{u-v}{4}}$, $\boxed{y = \frac{u+3v}{4}}$

Step 2.- Apply Theorem 9

15.9 Change of Vabs. Multiple Integrals

Theorem 9. – Consider a transf.

$$T: S \subset \mathbb{R}^2 \to R \subset \mathbb{R}^2$$

T: 

$$(u,v) \longrightarrow \begin{array}{l} x = x(u,v) \\ y = y(u,v) \end{array}$$

① T is C^1 conts. differentiable

② The Jacobian $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

is always of the same sign (never zero). on the interior of S.

then, T is one-to-one. on the interior of S.

③ R is bounded by a simple closed curve ∂R (It may be piecewise smooth).

④ f is conts on R.

Then,

$$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v))\, |J(u,v)|\, du\, dv.$$

<u>Applying theorem.</u>

$$J(u,v) = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} 1/4 & 1/4 \\ -1/4 & 3/4 \end{vmatrix} = \frac{3}{16} + \frac{1}{16} = \frac{1}{4}$$

Therefore,

$$\iint_R (4x+8y)\,dA = \iint_S 4\left(\frac{u-v}{4}\right) + 8\left(\frac{u+3v}{4}\right)\left(\frac{1}{4}\right) du\,dv$$

$$= \frac{1}{4}\int_{-4}^{4}\int_{0}^{8} \underbrace{(u-v+2u+6v)}_{3u+5v}\,du\,dv = \frac{1}{4}\int_{-4}^{4}\int_{0}^{8}(3u+5v)\,du\,dv$$

$v \quad u$

$$= \frac{1}{4}\int_{-4}^{4} \frac{3u^2}{4} + 5uv\Big|_0^8 dv = \frac{1}{4}\int_{-4}^{4}(96+40v)\,dv =$$

$$= \frac{1}{4}\left[96v + \frac{40v^2}{2}\right]\Big|_{-4}^{4} = \frac{1}{4}\left[96(4) + \frac{40(16)}{2} + 96(4) - \frac{40(16)}{2}\right]$$

$$= \underline{\underline{192}}.$$