MATH 214 Chapter 13

0.1 Definition 1: Limit of a vector function

Consider a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ defined on some interval I containing the point c, except possible at c itself. If

$$\lim_{t\to c} x(t) = l_1, \quad \lim_{t\to c} y(t) = l_2, \quad \lim_{t\to c} z(t) = l_3$$

exist then the limit of $\mathbf{r}(t)$ is defined as

$$\lim_{t\to c} \mathbf{r}(t) = \langle \lim_{t\to c} x(t), \lim_{t\to c} y(t), \lim_{t\to c} z(t)\rangle = \langle l_1, l_2, l_3\rangle$$

0.2 Definition 2: Continuity of a vector function

A vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is continuous at c if

$$\lim_{t\to c} \mathbf{r}(t) = \mathbf{r}(c)$$

0.3 Definition 3: Space curve

Consider a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ defined on some interval I, then the set $\mathcal{C}$ of all points $\{(x(t), y(t), z(t)) : t \in I\}$ is called a space curve.

Also, the equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad t \in I$$

are called parametric equations of $\mathcal{C}$ and t is called a parameter. Moreover, the vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is also called a parametrization of $\mathcal{C}$.

0.4 Definition 4: Derivative of a vector function

Consider a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. The derivative of $\mathbf{r}$ at $t = c$ is defined as the vector

$$\frac{d\mathbf{r}}{dt}(c) = \lim_{h\to 0} \frac{\mathbf{r}(c+h) - \mathbf{r}(c)}{h},$$

if this limit exists. The notation $\mathbf{r}'(c)$ is also used instead of $\frac{d\mathbf{r}}{dt}(c)$. If $\mathbf{r}(t)$ has derivatives for all t on (a, b), then $\mathbf{r}'(t)$ represents the derivative vector function of $\mathbf{r}(t)$ on (a, b).

0.5 Definition 4.2

If $\mathbf{r}'(t_0) \neq 0$ at t_0, then the vector

$$\mathbf{r}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0)\rangle$$

is called the tangent vector to the curve $\mathcal{C}$ defined by $\mathbf{r}(t)$ at the point $P_0(x_0, y_0, z_0)$

0.6 Definition 4.3

The line L

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\,\mathbf{r}'(t_0)$$

is called the tangent line to the graph of the curve parameterized by $\mathbf{r}(t)$ at the point $P_0(x_0, y_0, z_0)$.

0.7 Theorem 2: Differentiation Rules

If $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions in some interval J and a scalar function f is also differentiable in some interval I contained in J, then the following rules hold

$$\frac{d}{dt}\left(\mathbf{u}(\mathbf{t}) + \mathbf{v}(\mathbf{t})\right) = \frac{d\mathbf{u}}{dt}(t) + \frac{d\mathbf{v}}{dt}(t) \tag{1}$$

$$\frac{d(c\mathbf{u})}{dt}(t) = c\frac{d\mathbf{u}}{dt}(t) \tag{2}$$

$$\frac{d}{dt}\left(f(t)\mathbf{u}(\mathbf{t})\right)(t) = f'(t)\mathbf{u}(t) + f(t)\frac{d\mathbf{u}}{dt}(t) \tag{3}$$

$$\frac{d}{dt}\left(\mathbf{u}(t)\cdot\mathbf{v}(t)\right) = \frac{d\mathbf{u}}{dt}(t)\cdot\mathbf{v}(t) + \frac{d\mathbf{v}}{dt}(t)\cdot\mathbf{u}(t) \tag{4}$$

$$\frac{d}{dt}\left(\mathbf{u}(t)\times\mathbf{v}(t)\right) = \frac{d\mathbf{u}}{dt}(t)\times\mathbf{v}(t) + \frac{d\mathbf{v}}{dt}(t)\times\mathbf{u}(t) \tag{5}$$

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)) \tag{6}$$

0.8 Theorem 3: Chain rule

If for a *scalar function* f, it holds that $f(t)$ is in some interval I contained in the domain of $\mathbf{u}$, then u can be composed with f. If additionally, the vector function u is differentiable on its domain and the scalar function $\mathbf{f}$ is differentiable in I, then

$$\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)), \quad t \in I$$

0.9 Definition 5: Definite integral of a vector function

Consider a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ defined in $a \leq t \leq b$ where $x(t)$, $y(t)$, and $z(t)$ are continuous. The integral of $\mathbf{r}$ in $[a, b]$ is defined as the vector

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

0.10 Theorem 4: Arc length

Consider a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ defined in $a \leq t \leq b$ where $x(t)$, $y(t)$, and $z(t)$ are continuous differentiable on (a, b).

If the curve $\mathcal{C}$ corresponding to $\mathbf{r}(t)$ is traversed only once on $[a, b]$, then

$$\text{Arc length of } \mathcal{C} \text{ on } [a,b] = L = \int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

0.11 Definition 6: Arc length function

The arc length function $s(t)$ is defined as

$$s(t) = \int_a^t |\mathbf{r}'(t)|\,dt$$

Using the "fundamental theorem of calculus":

$$\frac{ds}{dt}(t) = |\mathbf{r}'(t)|$$

0.12 Definition 7: Smooth Curve

A curve $\mathcal{C}$ is smooth on I if it has a parametrization $\mathbf{r}(t)$ such that $\mathbf{r}'(t)$ is continuous on I.

0.13 Definition 8: Unit Tangent Vector

The unit tangent vector of a smooth curve at t is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

0.14 Definition 9: Smooth Curve

A curve $\mathcal{C}$ is smooth on I if it has a parametrization $\mathbf{r}(t)$ such that $\mathbf{r}'(t)$ is continuous on I.

0.15 Definition 10: Equivalent Expressions for Curvature

The curvature of a smooth curve at $P(x(t), y(t), z(t))$ can be expressed as

$$k(s) = \left|\frac{dT}{ds}\right|, \quad \text{the parameter } s \text{ is the arc length} \tag{7}$$

$$k(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \qquad k(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \tag{8}$$

0.16 Theorem 5

The unit tangent vector for a smooth curve $\mathcal{C}$ at $P(x(t), y(t), z(t))$ is orthogonal to its derivative. It means

$$\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$$

0.17 Definition 11: Unit Normal and Binormal Vectors

The unit normal vector for a smooth curve $\mathcal{C}$ at $P(x(t), y(t), z(t))$ is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

The binormal vector for a smooth curve $\mathcal{C}$ at $P(x(t), y(t), z(t))$ is given by

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

0.18 Definition 12: Osculating and Normal Planes

The osculating plane of a smooth curve at $P(x(t), y(t), z(t))$ is the plane that contains both the unit tangent vector $\mathbf{T}$ and the normal vector $\mathbf{N}$ and pass through P.

The normal plane of a smooth curve at $P(x(t), y(t), z(t))$ is the plane with normal vector $\mathbf{T}$