MATH 214 Chapter 14

Consider a function f such that

$$\begin{aligned} f : \mathcal{D} \subset \mathbb{R}^n &\to \mathbb{R} \\ (x_1, \ldots, x_n) &\to f(x_1, \ldots, x_n) \end{aligned} \tag{1}$$

0.1 Definition 1: Real-valued functions of n variables

A function of n variables is a rule that assigns a unique real number $f(x_1, \ldots, x_n)$ to each point $(x_1, \ldots, x_n)$ in $\mathcal{D}$

0.2 Definition 2: Domain and Range of functions of n variables

$f(x_1, \ldots, x_n)$ is called *the image of* $(x_1, \ldots, x_n)$ *under* f.

The set $\mathcal{D} \subset \mathbb{R}^n$ is called *the domain of* f.

The set of all real values $f(x_1, \ldots, x_n)$ is called *the Range of* f. It means

$$\text{range of } f = \{f(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in \mathcal{D}\}$$

.

Discuss examples of functions of two variables.

The graph of a real-valued function f of two is the variables is the set of points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y) \in \mathcal{D}$.

0.3 Definition 3: Level Curves for functions of two variables

Assume the graph of a function $z = f(x, y)$ describes a surface S in $\mathbb{R}^3$. For any real value c in the range of f, the intersection of of S with the plane $z = c$ is the curve

$$f(x, y) = c$$

The projection of this curve onto the xy-plane is called a level curve of f. It is also called a *contour curve.*

0.4 Definition 4: Limit of functions of two variables

Consider a function $f(x,y)$ defined on an open disc $B_r(a,b)$ centered at the point (a,b) with radius r, $B_r(a,b) = \{(x,y) : \sqrt{(x-a)^2+(y-b)^2} < r\})$, except possibly at (a,b), then

"the limit of f as (x,y) approaches (a,b) is L" or

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

if for all $\epsilon > 0$, there is a $\delta > 0$ such that
if $(x,y) \in B_r(a,b)$ and $0 < |(x,y)-(a,b)| = \sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y)-L| < \epsilon$

0.5 Theorem 1

If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path $\mathcal{C}_1$ and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path $\mathcal{C}_2$, and $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

0.6 Definition 5: Partial Derivatives

Let f be a real-valued function of two variables. The partial derivatives of f are the functions defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

and

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

provided that these limits exist.

Other notations are

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y)$$

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y)$$

0.7 Definition 6: Tangent plane

- Consider a function $f(x, y)$ defined in $\mathcal{D}$ that represents a surface S in $\mathbb{R}^3$.

- This function has continuous first partial derivatives in $\mathcal{D}$.

- Also, $P_0(x_0, y_0, z_0)$ is the center of an open disc $B_r(P_0) \subseteq \mathcal{D}$.

- Let C_1 and C_2 be the two curves obtained by intersecting the surface S with the two planes $x = x_0$ and $y = y_0$, respectively. If T_1 and T_2 are the tangent lines at the point P_0 to the curves C_1 and C_2, respectively,

then the tangent plane to the surface S at P_0 is defined to be the plane that contains both tangent lines T_1 and T_2.

0.8 Theorem 2: Equation of the tangent plane

If $f(x, y)$ and the surface S satisfy the conditions of the previous definition then, an equation of the tangent plane to the surface S, represented by the equation $z = f(x, y)$, at the point P_0 is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

0.9 Definition 7: Linear approximation of f at (x_0, y_0)

For f satisfying the conditions of Definition 1, we define the *linear approximation of* f *at* (x_0, y_0) as

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

0.10 Definition 8: Differentiability

Assume f has partial derivatives at $(x_0, y_0)) \in \mathcal{D}$, we will say that f is differentiable at (x_0, y_0) if

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1,\ \epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$

0.11 Theorem 3: Sufficient condition for differentiability

If the partial derivatives f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

0.12 Definition 8: Total differential

For f differentiable on $\mathcal{D}$, we define the total differential function as

$$dz(dx, dy) = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

In particular, if $dx = \Delta x = x - x_0$ and $dy = \Delta y = y - y_0$ then,

$$dz(\Delta x, \Delta y) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

0.13 Definition 9: Directional Derivatives

Consider a function $f(x, y)$ defined in $\mathcal{D}$ that represents a surface S in $\mathbb{R}^3$. Also, $P_0(x_0, y_0, z_0)$ is the center of an open disc $B_r(P_0) \subseteq \mathcal{D}$. Then, we define the directional derivative of f at (x_0, y_0) in the direction of the vector $\mathbf{u} = \langle a, b\rangle$ as

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

0.14 Theorem 4

If f is a differentiable function of x and y in a domain $\mathcal{D}$, then f has directional derivatives in the direction of any unit vectors $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

0.15 Definition 10: Gradient

If f has partial derivatives on $\mathcal{D}$, the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

0.16 Proposition 1

If f is a differentiable function of x and y in a domain $\mathcal{D}$, then the directional derivative of f in the direction of the unit vector $\mathbf{u} = \langle a, b\rangle$ can be written as

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

0.17 Remark 1

The previous definitions of linear approximations, differentiability, total differential, and directional derivatives are naturally extended to real-valued functions of three or more independent variables (see textbook). For example, for functions of three variables the gradient is defined as

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle.$$

The directional derivative in the direction of a vector $\mathbf{u} = \langle a,b,c\rangle$ is given by

$$D_{\mathbf{u}}f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c = \nabla f(x,y,z)\cdot\mathbf{u}$$

0.18 Remark 2

For a differentiable function f of two or more independent variables defined in a domain D consider all possible directional derivatives. Two important questions are:

i) In which direction does f change fastest?

ii) What is the maximum rate of change?

0.19 Theorem 5

If f is a differentiable function of two or more variables. The maximum value of the directional derivative of f at a point $\mathbf{x}_0$ is $|\nabla f(\mathbf{x}_0)|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x}_0)$.

0.20 Theorem 6

1. f is a differentiable function of three variables (x,y,z),
2. S is a surface with equation $f(x,y,z) = k$,
3. $P_0(x_0,y_0,z_0))$ is a point on S
4. C is a curve that lies on the surface S and passes through P_0.
5. C is parametrically represented by a differentiable function of t, $\mathbf{r(t)} = \langle x(t), y(t), z(t)\rangle$ and the point P_0 corresponds to the vector $\mathbf{r(t_0)} = \langle x(t_0), y(t_0), z(t_0)\rangle$

Then,

$$\nabla f(x_0,y_0,z_0)\cdot\mathbf{r}'(\mathbf{t_0}) = \mathbf{0}$$

0.21 Definition 11: Tangent plane to a level surface S.

For a differentiable function f on D with $P_0(x_0, y_0, z_0)$ in D and $\nabla f(x_0, y_0, z_0) \neq 0$, we define *the tangent plane to the level surface* $f(x, y, z) = k$ at P_0 as the plane that passes through P_0 and has normal vector $\nabla f(x_0, y_0, z_0)$. Then, the plane is given by the equation

$$f_x(x_0, y_0, z_0)(x - x_o) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

Specialize this theorem to functions of two variables $f(x, y)$.
What can be said?

0.22 Definition 12: Local Maximum and Local Minimum

Let $\mathcal{D} \subseteq \mathbb{R}^2$ be the domain of the function f.

i) A function of two variables has a **local minimum at** (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in some $\mathcal{B}_r(a, b)$ contained in $\mathcal{D}$. The number $f(a, b)$ is called a **a local minimum value of** f.

ii) A function of two variables has a **local maximum at** (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some $\mathcal{B}_r(a, b)$ contained in $\mathcal{D}$. The number $f(a, b)$ is called **a local maximum value of** f.
Discuss the concepts of absolute maximum and minimum.

0.23 Theorem 7: Necessary Condition for a Relative Extrema

If the following two conditions are verified:
i) f has a local maximum or minimum at (a, b)
. and
ii)The first order partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist, then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

0.24 Definition 13: Critical Points

A point (a, b) is called a critical point of a function f if

$$f_x(a, b) = 0 \qquad \text{and} \qquad f_y(a, b) = 0,$$

or if one of these partial derivatives does not exist at (a, b).

0.25 Theorem 8: Test for Relative Extrema

Consider a function f defined on a domain $\mathcal{D}$ of $\mathbb{R}^2$, then if

i) There is a point (a, b) in $\mathcal{D}$, such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. It means the point (a, b) is a critical point of f.

ii) The second-order partial derivatives f_{xx}, f_{xy}, and f_{yy} are continuous in a disk $\mathcal{B}_r(a, b)$ contained in $\mathcal{D}$.

iii) $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$,

then

a) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f reaches a local minimum $f(a, b)$ at the critical point (a, b).

b) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f reaches a local maximum $f(a, b)$ at the critical point (a, b).

c) If $D(a, b) < 0$, then $f(a, b)$ is not a relative extrema and there is a **saddle point on the graph of** f at the critical point (a, b).

d) If $D(a, b) = 0$ the test gives no information.

Why?
Consider $D_{\mathbf{u}}f$ in the direction of $\mathbf{u} = \langle h, k\rangle$, then $D_{\mathbf{u}}f = f_x h + f_y k$ and

$$D^2_{\mathbf{u}}f = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f^2_{xy})$$

0.26 Theorem 9: Conts. functions on closed and bounded sets

If f is continuous in a closed and bounded set D in $\mathbb{R}^2$, then f reaches an absolute maximum value and an absolute minimum value at some points (x_1, y_1) and (x_2, y_2) in D, respectively. *Discuss procedure to obtain them.*

0.27 Procedure to find Absolute Maximum and Abs. Minimum

Assume f is continuous in a closed and bounded set D in $\mathbb{R}^2$ with the closed curve $\mathcal{C}$, defined by the parametric equations $(x(t), y(t))$ $t \in [a, b]$, as a boundary. Then to find the absolute maximum and absolute minimum values do the following:

1. Find all critical points (x_i, y_i) of f in $\mathcal{D}$.
2. Consider the function f restricted to the boundary curve $\mathcal{C}$. Then, f reduces to a function of only one variable t. In fact, $g(t) = f(x(t), y(t))$, along $\mathcal{C}$. This function g is also continuous in $[a, b]$.
3. Find all critical points t_j of g in (a, b).
4. Evaluate f at all critical points (x_i, y_i) and $(x(t_j), y(t_j))$. The largest of these values is the absolute maximum value of f in $\mathcal{D}$ and the smallest is the minimum value.

0.28 Theorem 10: Lagrange Multipliers

a) If f is continuously differentiable on D
b) The function g defined on $H \subseteq D$ is also continuously differentiable.
c) f, subject to the constraint $g(\mathbf{x}) = 0$, has a maximum or minimum value at $\mathbf{x_0}$.
d) $g(\mathbf{x_0}) \neq 0$, then
There is a $\lambda \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x_0}) = \lambda \nabla g(\mathbf{x_0})$$

Procedure to find maximum and minimum values for a function f of three variables x, y, z, subject to a constraint $g(x, y, z) = k$.

1. Find all points (x, y, z) and any value λ that satisfies the nonlinear system of equations given by the four equations

$$\nabla f(x, y, z) = \lambda g(x, y, z) \tag{2}$$
$$g(x, y, z) = k \tag{3}$$

2. Evaluate f at all the points obtained in the previous step. The largest of these values is the maximum value of f; the smallest is the minimum value. In case that there is only one point (x_0, y_0, z_0) in $\mathcal{D}$ from the previous step, we can choose an arbitrary point (a, b, c) in $\mathcal{D}$ satisfying the constraint $g(a, b, c) = k$ and compare the values of f at these two points. If $f(a, b, c) \leq f(x_0, y_0, z_0)$ then $f(x_0, y_0, z_0)$ is the maximum of f in $\mathcal{D}$, otherwise is the minimum.