MATH 214

Chapter 16: Vector Calculus

0.1 Line Integrals

Consider a smooth plane curve $\mathcal{C}$ in the space given by the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t), \qquad a \leq t \leq b \tag{1}$$

or $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then, $\mathbf{r}$' is continuous and $\mathbf{r}'(t) \neq 0$.

Construct a uniform partition of $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ with a point t_i^* in it. Show Fig 1 book for corresponding partition along arc length parameter s.

0.2 Definition 0.1: Line integral of Scalar Function f along $\mathcal{C}$

- f is defined on smooth curve $\mathcal{C}$ given by equations (1)
Then,

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

is the line integral of f along $\mathcal{C}$, if this limit exists.

0.3 Theorem 0.1: Evaluation of Line Integral of a Scalar Function f along $\mathcal{C}$

Discuss: Evaluation of arc length of curve $\mathcal{C}$ between a and b.

The line integral of f along $\mathcal{C}$ can be evaluated as

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

0.4 Theorem 0.2: Line Integral of a Scalar Function f along $\mathcal{C}$ with respect to x, y, and z

$$\int_{\mathcal{C}} f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\,x'(t)\,dt$$
$$\int_{\mathcal{C}} f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))\,y'(t)\,dt$$
$$\int_{\mathcal{C}} f(x,y,z)\,dz = \int_a^b f(x(t),y(t),z(t))\,z'(t)\,dt$$

0.5 Definition 0.2: Vector Field

If $E \subseteq \mathbb{R}^3$, then a vector field on $\mathbb{R}^3$ is a function F that assigns to each point (x,y,z) in E a three-dimensional vector $\mathbf{F}(x,y,z)$. It ca be expressed as

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

0.6 Definition 0.3: Flow Lines or Streamlines

The flow lines of a vector field $\mathbf{F}$ are the curves $\mathcal{C}$ in the space (or plane) such that the vectors in the vector field are tangents to these curves.

Alternative Definition:
The flow lines or streamlines of a vector field are the paths followed by particles whose velocity field is the given vector field.

More precisely, if

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

and a flow line curve $\mathcal{C}$ has the parametric representation $\mathbf{r}(t) = (x(t),y(t),z(t))$ then the components of $\mathbf{r}$ satisfy the differential equation

$$\frac{dx}{dt}(t) = P(x(t),y(t),z(t)) \quad \frac{dy}{dt}(t) = Q(x(t),y(t),z(t)) \quad \frac{dz}{dt}(t) = R(x(t),y(t),z(t))$$

0.7 Definition 0.4: Work Done to Move Particle with Force F along $\mathcal{C}$

$$W = \int_{\mathcal{C}} \mathbf{F}(x,y,z)\cdot\mathbf{T}(x,y,z)\,ds = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds$$

where, $\mathbf{T}(x,y,z)$ is the unit tangent vector at the point (x,y,z) on $\mathcal{C}$.

0.8 Theorem 0.3: Work Done to Move Particle with Force F along $\mathcal{C}$, Using a parametric Representation of $\mathcal{C}$

$$W = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

Show Why?

0.9 Definition 0.5: Line Integral of a Vector Field F along $\mathcal{C}$

- $\mathbf{F}$ is a continuous vector field defined on $\mathcal{C}$
- $\mathcal{C}$ is a smooth curve given by $\mathbf{r}(t)$, $a \leq t \leq b$

Then, the line integral of the vector field $\mathbf{F}$ along $\mathcal{C}$ is given by

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$$

0.10 Alternative Representation of a Line Integral of a Vector Field F along $\mathcal{C}$

- If $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$
- $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$

Then,

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_{\mathcal{C}} P(x, y, z)\, dx + \int_{\mathcal{C}} Q(x, y, z)\, dy + \int_{\mathcal{C}} R(x, y, z)\, dz$$

Show Why?

0.11 Theorem 1: Integration of Conservative Vector Fields

– C is a smooth curve given by $\mathbf{r}(t)$ where $a \leq t \leq b$,
. $\mathbf{r}(a) = \langle x_1, y_1, z_1 \rangle$, and $\mathbf{r}(b) = \langle x_2, y_2, z_2 \rangle$
– f is defined on a domain D containing C,
– f is differentiable and its gradient vector ∇f is continuous on C. Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Work on proof.

0.12 Alternative for Theorem 1: Integration of Conservative Vector Fields

– $\mathbf{F}$ is continuous on $D \subseteq \mathbb{R}^3$,
– D contains a smooth curve C given by $\mathbf{r}(t)$ where $a \leq t \leq b$,
. $\mathbf{r}(a) = \langle x_1, y_1, z_1 \rangle$, and $\mathbf{r}(b) = \langle x_2, y_2, z_2 \rangle$
– $\mathbf{F}$ is a conservative vector field in the domain D. It means there is f such that $\mathbf{F} = \nabla f$.
Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

Work on proof and discuss examples

0.13 Definition 1: Independence of Path

– $\mathbf{F}$ continuous vector field on D.
– C_1 and C_2 two curves or paths contained in D.
– C_1 and C_2 have the same initial and terminal point.
Then, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

0.14 Theorem 2:

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise-smooth closed path C in D.
Discuss proof.
A closed path is one for which its terminal point coincides with its initial point.

0.15 Corollary:

If $\mathbf{F}$ is a conservative vector field defined on D then, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D.

Is the reciprocal statement true?

0.16 Definition 2: Open and Connected Sets

– D in $\mathbb{R}^{2,3}$ is open if for every point P in D there is a disk (or ball) with center on P that is contained in D.
– D is a connected set if any two points in D can be joined by a curve contained in D.

0.17 Theorem 3

– $\mathbf{F}$ is a continuous vector field on an open connected region D.
– $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D.

Then $\mathbf{F}$ is a conservative vector field on D (there is f such that $\nabla f = \mathbf{F}$)

0.18 Theorem 4: Property of a Conservative Vector Field.

– $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a conservative vector field on D.
– P and Q have continuous first-order partial derivatives on the domain D.

Then, for all (x, y) in D

$$P_y(x, y) = Q_x(x, y)$$

work on proof

0.19 Theorem 5: Easy Way to Identify a Conservative Vector Field.

– $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a vector field defined on an open and simply connected region D.
– P and Q have continuous first-order partial derivatives on the domain D.

If $P_y(x, y) = Q_x(x, y)$ for all (x, y) in D, then, $\mathbf{F}$ is conservative.

How can we find the potential f corresponding to a conservative vector field?
Show examples

0.20 Theorem 6: Green's Theorem

– C is positively oriented, piecewise smooth, simple closed curve in the plane and is the boundary of a region D.
– P and Q have continuous first order partial derivatives on an open region that contains D. Then,

$$\int_C P\,dx + Q\,dy = \int\int_D (Q_x - P_y)\ dA$$

0.21 Corollary 1: Green's Theorem for Regions with a Hole

– If D is a region enclosed by two simple and piecewise smooth curves C_1 and C_2
– C_2 is contained in the region enclosed by C_1.
– C_1 is positively oriented (counterclockwise for this case), and C_2 has the same orientation as C_1
– P and Q have continuous first order partial derivatives on an open region that contains D.
Then,

$$\int\int_D (Q_x - P_y)\ dA = \int_{C_1} P\,dx + Q\,dy - \int_{C_2} P\,dx + Q\,dy$$

0.22 Corollary 2: Line Integral Over Complex Curves

– If D is a region enclosed by two simple piecewise and smooth curves C_1 and C_2
– C_2 is contained in the region enclosed by C_1.
– C_1 is positively oriented (counterclockwise for this case), and C_2 has the same orientation as C_1
– P and Q have continuous first order partial derivatives on an open region that contains D.
– $P_y(x, y) = Q_x(x, y)$ on the open region that contains D. Then,

$$\int_{C_1} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy$$

0.23 Definition 3: Parametric Surfaces

– Consider a region D in the uv-plane
– and functions $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ defined on D.
Then, the vector-valued function defined on D as

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle \tag{2}$$

is called a parametric equation of the parameters u and v and the set of points $\{(x(u, v, y(u, v), z(u, v) : (u, v) \in D\}$ is called a parametric surface S corresponding to the vector function $\mathbf{r}$.

0.24 Definition 4: Normal Vector to a Parametric Surface

– Consider a parametric surface S given by (2) defined on D.
– and $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ have partial derivatives on D and

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\rangle \times \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\rangle \neq 0, \qquad \text{on} \quad D \tag{3}$$

Then, the surface S is smooth and the normal vector to its tangent plane at $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$ is defined as $\mathbf{r}_u \times \mathbf{r}_v(u_0, v_0)$

0.25 Definition 5: Surface Area of S

– If S is smooth parametric surface given by (2).
Then, the surface area of S is defined as

$$A(S) = \int\int_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA \tag{4}$$

0.26 Theorem 7: Surface Area of the Graph of $z = f(x, y)$

– For $z = f(x, y)$, we can define the corresponding parametric equations as $x = x$ $y = y$, and $z = f(x, y)$
– If f has continuous partial derivatives, then the surface area of S is given by

$$A(S) = \int\int_D \sqrt{1 + z_x^2 + z_y^2}\, dA.$$

0.27 Definition 6: Surface Integral of a Scalar Function f

– If f is continuous on a region R containing the surface S.
– S is a smooth surface and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonparallel in D.
Then, the surface integral of f over the surface S is defined as

$$\int\int_S f(x,y,z)\,dS = \int\int_D f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\,dA \tag{5}$$

In particular, if S is defined as $z = g(x,y)$, then

$$\int\int_S f(x,y,z)\,dS = \int\int_D f(x,y,g(x,y))\,\sqrt{z_x^2 + z_y^2 + 1}\,dA \tag{6}$$

Work on Example 3 of the book

0.28 Definition 7: Surface Orientation

A surface S for which there is a continuously varying unit vector $\mathbf{n}$ is called an oriented surface. The vector $\mathbf{n}$ provides an orientation.

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For a closed surface S enclosing a solid E, positive orientation is defined as the one for which the normal vectors point out from E.

0.29 Definition 8: Surface Integral of Vector Fields

– F is a continuous vector field defined on oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of F over S is defined as

$$\int\int_S \mathbf{F}\cdot\mathbf{S} = \int\int_S \mathbf{F}\cdot\mathbf{n}\,dS \tag{7}$$

It is also called flux of $\mathbf{F}$ across S.
If S is given by the parametric function $\mathbf{r}(u,v)$ then,

$$\int\int_S \mathbf{F}\cdot\mathbf{S} = \int\int_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA \tag{8}$$

Show this and work on Example 5 if time permits

0.30 Theorem 8: Stoke's Theorem

– S is an oriented piecewise-smooth surface, bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation.
– F is a vector field whose components have continuous partial derivatives on a region R that contains the surface S. Then,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S curl \ \mathbf{F} \cdot d\mathbf{S} \tag{9}$$

Work on Example 1

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Green's theorem is special case of Stoke's Theorem. In fact,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_S curl \ \mathbf{F} \cdot d\mathbf{S} = \int \int_S curl \ \mathbf{F} \cdot \mathbf{k} \ d\mathbf{A} \tag{10}$$

0.31 Theorem 9: The Divergence or Gauss Theorem

– E is a simple solid region with surface boundary S.
– S has positive (outward) orientation.
– F is a vector field whose component functions have continuous partial derivatives on an open region R containing E. Then,

$$\int \int \int_E div \ \mathbf{F} \, dV = \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_S \mathbf{F} \cdot \mathbf{n} \ dS, \tag{11}$$

where $\mathbf{n}$ is the unit outward normal vector to S

Work on Example 2