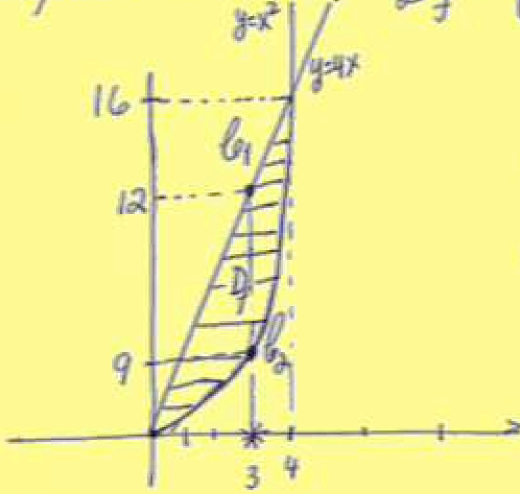
14.7.

Example: Consider the function

$$f(x,y) = (x-3)^2 + y^2$$

defined over $D_f = \{(x,y): 0 \le x \le 4, \ x^2 \le y \le 4x\}$



$C_1: \hat{r}_1(t) = (t, 4t), \quad t \in [0,4].$

$C_2: \hat{r}_2(t) = (t, t^2), \quad t \in [0,4].$

Find global max and global minimum for $f(x,y)$.

The function $f(x,y)$ is differentiable at every interior point in domain D_f. Therefore, the only critical points are either stationary points or boundary points.

Stationary points:

$$\nabla f(x,y) = (2(x-3), 2y) = 0 \quad \text{iff} \quad \begin{matrix} x=3 \\ y=0 \end{matrix}$$

But this point $(3,0) \notin D_f$. Therefore, the only possibility for global max or global min is over the boundary $C_1 \cup C_2$.

Global max and Global min over boundary C_1

$$f_1(t) = f(\vec{r}_1(t)) = f(t,4t) = (t-3)^2 + 16t^2 = 17t^2 - 6t + 9, \quad t \in [0,4]$$

$$\Rightarrow f_1'(t) = 34t - 6 = 0 \Rightarrow t = \frac{6}{34} = \frac{3}{17}$$

Thus, $t^* = \frac{3}{17}$ is an stationary point over the boundary $\mathcal{C}_1$ (straight line). $\longrightarrow$ The corresponding point in $\mathcal{D}$ for $t^* = 3/17$ is $(3/17, 12/17)$

Also, $f_1''(t) = 34 > 0 \Rightarrow t^* = \frac{3}{17}$ is a local minimum

and $f_1(3/17) = f(3/17, 12/17) \simeq 8.47.$

To find global minimum and maximum, we need to Consider also the extreme of the intervals $[0,4]$

In fact,

$$f_1(0) = f(0,0) = 9$$

$$f_1(4) = f(4,16) = 1^2 + 16^2 = 257.$$

Therefore, Global maximum over $\mathcal{C}_1$: at $(4,16)$

$f(4,16) = 257.$

Global minimum over $\mathcal{C}_1$: at $(3/17, 12/17)$

$f(3/17, 12/17) \simeq 8.47.$

Global maximum and global minimum over boundary $\mathcal{C}_2$

$$f_2(t) = f(\vec{r}_2(t)) = f(t, t^2) = (t-3)^2 + t^4 = t^4 + t^2 - 6t + 9, \quad t \in [0,4]$$

$$\Rightarrow f_2'(t) = 4t^3 + 2t - 6 = 0 \Leftrightarrow (t-1)(4t^2 + 4t + 6) = 0$$

$t = 1$ only real root.

Since, $f_2''(t) = 12t + 2 \Big|_{t=1} = 14 > 0$

then $\hat{t} = 1$ is a local minimum

and $f_2(1) = f(1,1) = 4 + 1 = 5$

To find global max and global min over C_2, we need to consider also the ends of the interval $[0,4]$.

In fact,

$$f_2(0) = f(0,0) = 9$$

$$f_2(4) = f(4,16) = 1 + 16^2 = 257$$

Therefore,

Global maximum over C_2: at $(4,16)$ $f(4,16) = 257$

Global minimum over C_2: at $(1,1)$ $f(1,1) = 5$

Finally, Global maximum of $f(x,y)$ over D_f: at $(4,16)$ $f(4,16) = 257$.

Global minimum of $f(x,y)$ over D_f: at $(1,1)$ $f(1,1) = 5$.