15.8 Spherical Coordinates.

a) Description of sphere:

$$x^2+y^2+z^2=1 \implies \rho^2=1 \Rightarrow \rho=1$$

b) Description of Cone:

$$z=\sqrt{3x^2+3y^2} \implies \rho\cos\phi=\sqrt{3}\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}$$

$$\text{or } \rho\cos\phi=\sqrt{3}\sqrt{\rho^2\sin^2\phi\,\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}}=$$

$$=\sqrt{3}\sqrt{\rho^2\sin^2\phi}=\sqrt{3}\,\rho\sin\phi$$

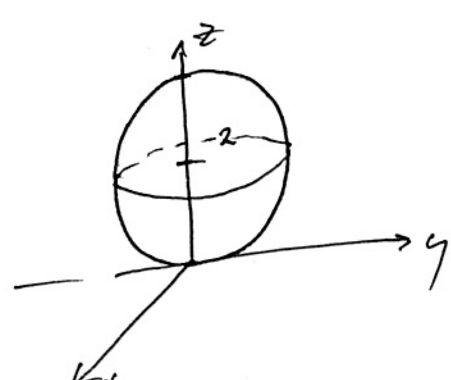
or Cone: $\cancel{\rho}\cos\phi=\sqrt{3}\sin\phi\,\cancel{\rho}$

or $\tan\phi=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3} \implies \boxed{\phi=\pi/6}$

c) Description of sphere:

$$\boxed{x^2+y^2+z^2=2z} \implies \rho^2=2\rho\cos\phi$$

$$\text{or } \boxed{x^2+y^2+(z-2)^2=4} \qquad \boxed{\rho=2\cos\phi}$$



Integrals over regions E in spherical coords.

(A) Spherical wedges:

$$E = \{ (\rho, \theta, \phi) \mid 0 \le \rho \le a, \ \alpha \le \theta \le \beta; \ c \le \phi \le d \}$$

$$\iiint_E f(x,y,z)\, dV =$$

$$= \int_{\phi=c}^{d} \int_{\theta=\alpha}^{\beta} \int_{0}^{a} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

(B) More general regions:

$$E = \{ (\rho, \theta, \phi) \mid \alpha \le \theta \le \beta, \ c \le \phi \le d, \ g_1(\theta,\phi) \le \rho \le g_2(\theta, \phi) \}$$

$$\iiint_E f(x,y,z)\, dV =$$

$$= \int_{\theta=\alpha}^{\beta} \int_{\phi=c}^{d} \int_{\rho=g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Spherical Wedge

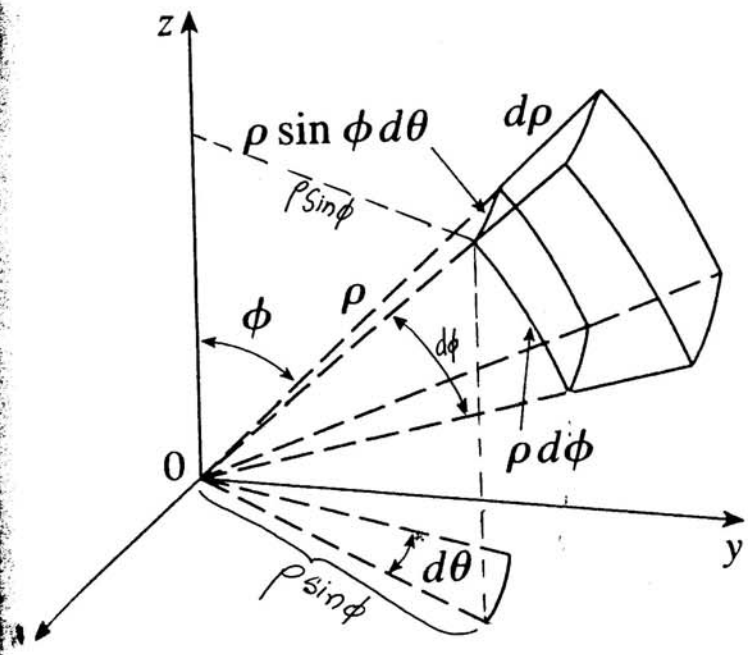


FIGURE 8

Volume element in spherical coordinates: $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

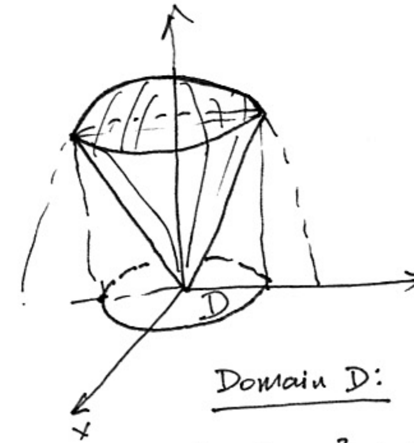
$$\Delta V_k \simeq (\rho_k \Delta\phi_k)(\rho_k \sin(\phi_k)\Delta\theta_k)\Delta\rho_k = \rho_k^2 \sin\phi_k \Delta\rho_k \Delta\theta_k \Delta\phi_k$$

Using MVT, it can be shown that there is a point $(\rho_k^*, \theta_k^*, \phi_k^*) \in V_k$ Satisfying

$$\Delta V_k = \rho_k^{*2} \sin(\phi_k^*) \Delta\rho_k \, \Delta\theta_k \, \Delta\phi_k \,. \text{ exactly}$$

15.8 Spherical Coords

(A) Ice-cream Cone



Vol. inside:

Cone: $z = \sqrt{3x^2+3y^2}$

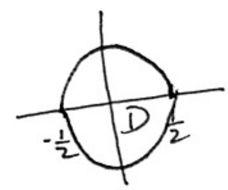
Sphere: $x^2+y^2+z^2=1$

Combining $\longrightarrow$ $x^2+y^2+3x^2+3y^2=1$

Domain D:

D: $x^2+y^2=\frac{1}{4}$

Circle of radius $\frac{1}{2}$.



I) In Cartesian:

$$\int_{-1/2}^{1/2}\int_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}}\int_{\sqrt{3x^2+3y^2}}^{+\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

$$= \int_{-1/2}^{1/2}\int_{-\sqrt{1/4-x^2}}^{\sqrt{1/4-x^2}} \left(\sqrt{1-x^2-y^2}-\sqrt{3x^2+3y^2}\right) dy\,dx = ?$$

Hard to integrate!

II) In Spherical coords ρ, θ, ϕ

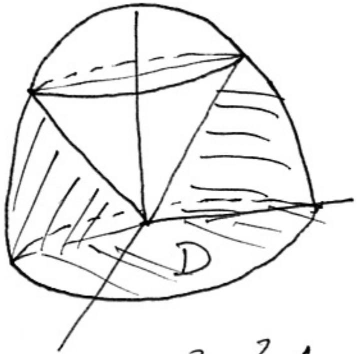
$$E \equiv \left\{(\rho,\theta,\phi) \,/\, 0\le\rho\le 1,\ 0\le\theta\le 2\pi,\ 0\le\phi\le \pi/6\right\}$$

$$\text{Vol.} = \iiint_E dV = \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/6} \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho \underset{\text{algebra}}{\overset{\text{Easy}}{=\ldots}} = \frac{\pi}{3}\left(2-\sqrt{3}\right).$$

(B) Vol. below Ice-cream cone and inside Sphere

$$\text{Vol} = \int_0^1 \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$\rho \quad \theta \quad \phi$



D: $x^2+y^2=1$

$$= \int_0^1 \int_0^{2\pi} -\rho^2 \cos\phi \Big|_{\pi/6}^{\pi/2} d\theta \, d\rho =$$

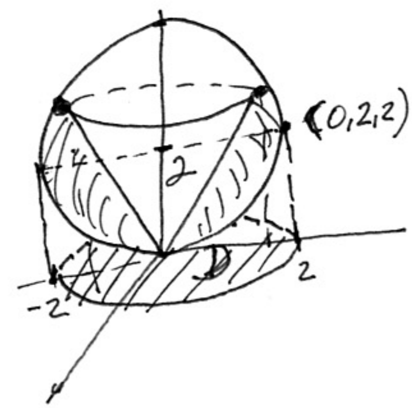
$$= -\int_0^1 \int_0^{2\pi} \rho^2 \left(-\frac{\sqrt{3}}{2}\right) d\theta \, d\rho = \int_0^1 \cancel{2}\pi \frac{\sqrt{3}}{\cancel{2}} \rho^2 d\rho = \pi\sqrt{3} \, \frac{\rho^3}{3}\Big|_0^1 =$$

$$= \frac{\pi\sqrt{3}}{3}$$

15.8 Spherical Coords. Triple Integrals. (Cont.)

ⓒ Vol of Solid below Cone: $z=\sqrt{3x^2+3y^2}$ and inside Sphere: $x^2+y^2+z^2=2z$.

I) In Cartesian coords.



Domain D:

Sphere: $x^2+y^2+(z-2)^2=4$

when $z=2$ the projection gives us the domain D. $\longrightarrow z=\pm\sqrt{4-x^2-y^2}+2$

$D: x^2+y^2=4$

Then,

$$I=\iint_{\substack{D\\ xy\text{-plane}}}\int_{-\sqrt{4-x^2-y^2}+2}^{\sqrt{3x^2+3y^2}} dz\,dy\,dx+\iint_D\int_{+\sqrt{4-x^2-y^2}+2}^{\sqrt{3x^2+3y^2}} dz\,dy\,dx.$$

II) In Spherical Coords.

Cone:

a) $z=\sqrt{3x^2+3y^2}$

$\rho\cos\phi=\sqrt{3}\sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}=\sqrt{3}\,\rho\sin\phi$

or

$\cos\phi=\sqrt{3}\sin\phi$

$\Rightarrow \tan\phi=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3} \Rightarrow \boxed{\phi=\pi/6\ (30^\circ)}$

Equ. of cone.

b) Sphere : $x^2+y^2+z^2=4z$

$$\rho^2 = 4\cancel{\rho}\cos\phi \Rightarrow \boxed{\rho = 4\cos\phi},\quad 0 \le \phi \le \pi/2 .$$

then,

$$I = \iiint_E dV = \int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^{4\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$\theta \quad \phi \quad \rho$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/2} \left.\frac{\rho^3}{3}\right|_0^{4\cos\phi} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/2} \frac{64}{3}\cos^3\phi \sin\phi \, d\phi \, d\theta =$$

$$= -\int_0^{2\pi} \frac{64}{3} \left.\frac{\cos^4\phi}{4}\right|_{\pi/6}^{\pi/2} d\theta = -\int_0^{2\pi} \frac{16}{3}\left(-\left(\frac{\sqrt{3}}{2}\right)^4\right) d\theta =$$

$$= \int_0^{2\pi} \left(\frac{\cancel{16}}{3}\frac{9}{\cancel{16}}\right) d\theta = \frac{9}{3}(2\pi) = \frac{18\pi}{3}$$

Spherical Wedge

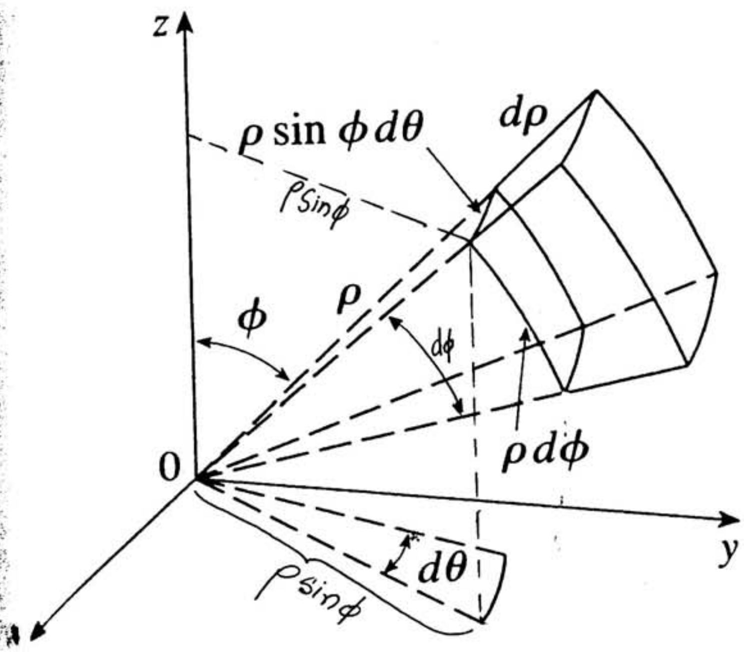


FIGURE 8
Volume element in spherical coordinates: $dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$

$$\Delta V_k \simeq (\rho_k \Delta\phi_k)(\rho_k \sin(\phi_k) \Delta\theta_k) \Delta\rho_k = \rho_k^2 \sin\phi_k \, \Delta\rho_k \, \Delta\theta_k \, \Delta\phi_k$$

Using MVT, it can be shown that there is a point $(\rho_k^*, \theta_k^*, \phi_k^*) \in V_k$ satisfying

$$\Delta V_k = \rho_k^{*2} \sin(\phi_k^*) \Delta\rho_k \, \Delta\theta_k \, \Delta\phi_k \text{. exactly}$$