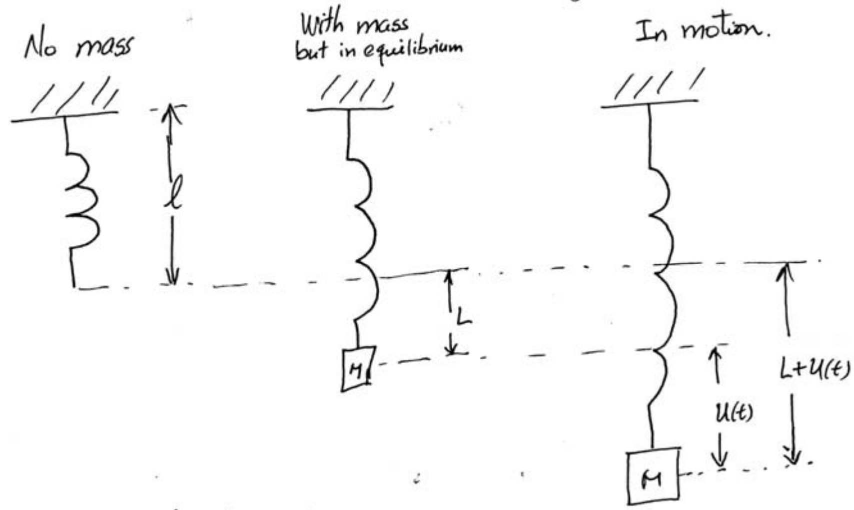
3.8 Mechanical Vibrations.

Motion of a mass on a Vibrating Spring:



l: Original length of spring.

L: Elongation from original length due to the mass weight.

$u(t)$: Displacement from equilibrium position at time t.

Mathematical Model

Based on Newton 2nd Law:

$$m \frac{d^2u}{dt^2}(t) = \sum \text{Forces.}$$
$$= F_d + F_s + mg + F_e(t).$$

where

F_d: Resistance, damping, drag or viscous force.

F_s: Spring restoring force.

mg: gravity

F_e: External force.

Appropriate expresions for F_s and F_d.

F_s is always (if elongation L of spring is small)

(I) Proportional to the elongation $L+u(t)$ from original spring length.

(II) Acting to restore spring to its original or natural position.

Then,

a) if $\quad L+u>0 \Rightarrow F_s<0 \overset{\uparrow up}{\Rightarrow} F_s=-\overset{>0}{K}\,\overset{>0}{(L+u)}<0$

b) if $\quad L+u<0 \Rightarrow F_s>0 \underset{\downarrow down}{\Rightarrow} F_s=-\overset{>0}{K}\,\overset{<0}{(L+u)}>0$

Therefore, in any case

$$\boxed{F_s=-K\,(L+u).} \quad \text{Hooke's Law.}$$

K: Spring constant

F_d is always

(I) acting in the direction opposite to the direction of motion.

(II) (Linear approximation) proportional to the speed $|u'(t)|$.

Therefore,

a) if $u'(t) > 0 \Leftrightarrow$ Mass is moving downwards $\Rightarrow$ F_d is upwards $F_d < 0$.

$$F_d = -\overset{>0}{\gamma}\,\overset{>0}{u'(t)} < 0$$

b) if $u'(t) < 0 \Leftrightarrow$ Mass is moving upwards $\Rightarrow$ F_d is downwards. $\Rightarrow F_d > 0$

$$F_d = -\overset{>0}{\gamma}\,\overset{<0}{u'(t)} > 0.$$

Thus, in any case

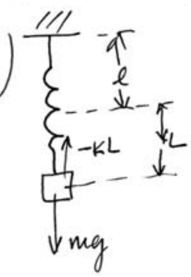
$$\boxed{F_d = -\gamma u'(t)}$$

<u>Final model:</u>

$$m\frac{d^2u}{dt^2} = -\gamma\frac{du}{dt} - k(L+u) + mg + F_e(t).$$

$$= -\gamma\frac{du}{dt} - kL - ku + mg + F_e(t)$$

Now, At the equilibrium position (Mass is at rest)



$m u''(t) = 0.$

Newt. 2nd Law: $0 = mg - kL \Rightarrow mg = kL$

Then, the model reduces to

$$m\frac{d^2u}{dt^2} = -\gamma\frac{du}{dt} - \cancel{kL} - ku + \cancel{mg} + F_e(t)$$

or

$$\boxed{m\frac{d^2u}{dt^2}(t) + \gamma\frac{du}{dt} + ku(t) = F_e(t)}$$

Linear, 2^{nd} order, const. coeffs., nonhomog. equation

Easy!

We will be interested in the study of the behavior of the solutions.