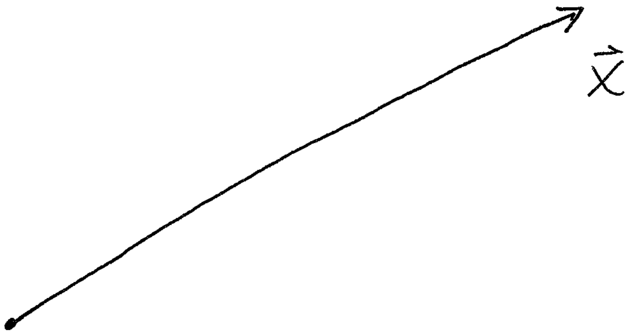
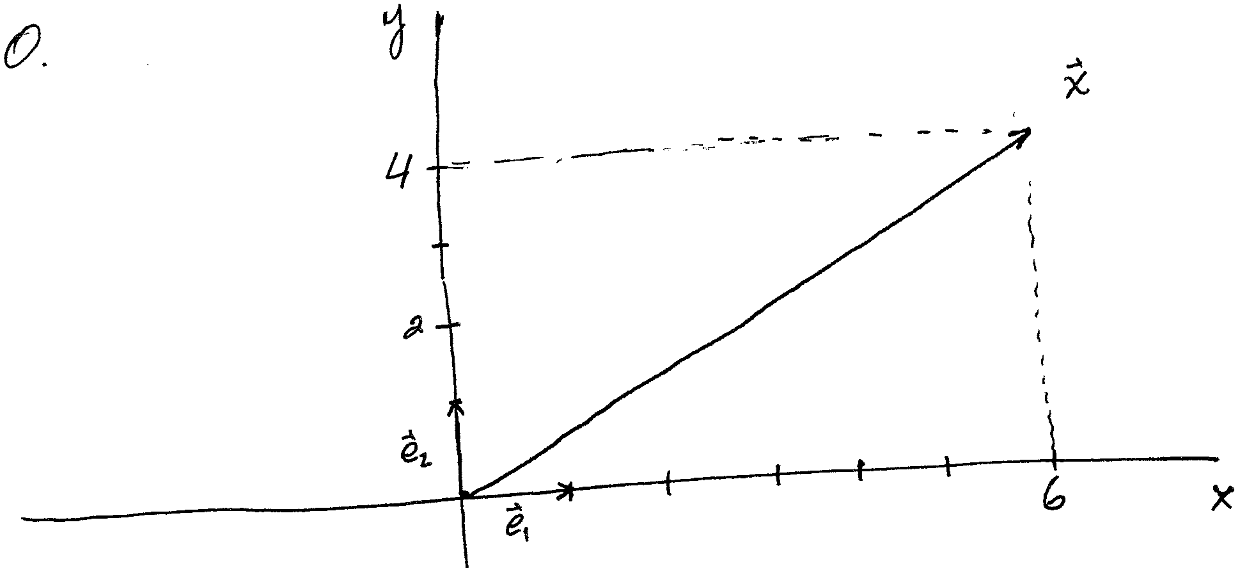
CHANGE OF BASIS

(I) Arrow in the plane with a given magnitude and a given length.



(II) Define an origin "O" at the tail of $\vec{x}$ and introduce two orthogonal axis passing through O.



Define two unit vectors
$\vec{e}_1$ along x-axis
$\vec{e}_2$ " y-axis

projection of $\vec{x}$ on x-axis = 6
projection of $\vec{x}$ on y-axis = 4

Parallelogram rule:

$$\boxed{\vec{x} = 6\vec{e}_1 + 4\vec{e}_2}$$

The set $E = \{\vec{e}_1, \vec{e}_2\}$ form a basis in $\mathbb{R}^2$.

We define the column vector $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$ as the coordinates of the Vector $\vec{x}$ with respect to the basis E.

Notation: $$[\vec{x}]_E = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

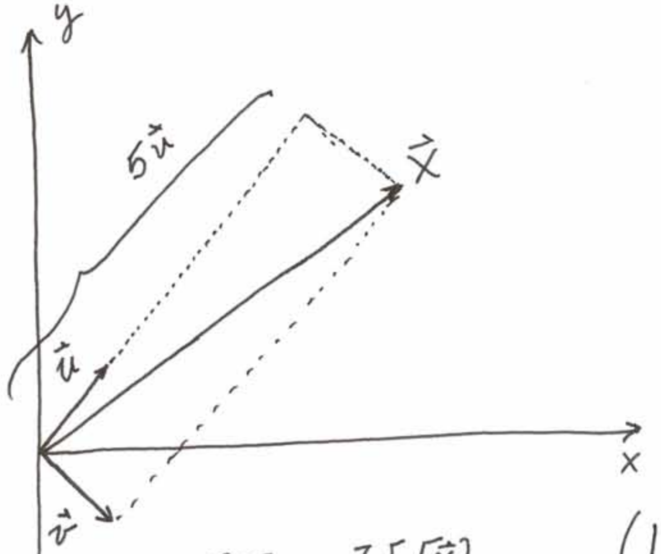
Ⅲ Consider a new basis $F = \{\vec{u}, \vec{v}\}$, where

$[\vec{u}]_E = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is equivalent to $\vec{u} = \vec{e}_1 + \vec{e}_2$

$[\vec{v}]_E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which is equivalent to $\vec{v} = \vec{e}_1 - \vec{e}_2$

Parallelogram rule:

$\vec{x} = 5\vec{u} + \vec{v} \Leftrightarrow [\vec{x}]_F = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$



Also,

$\vec{x} = 5\vec{u} + \vec{v} = 5(\vec{e}_1 + \vec{e}_2) + (\vec{e}_1 - \vec{e}_2)$

$= (5+1)\vec{e}_1 + (5-1)\vec{e}_2$

Which is equivalent to

$$[\vec{x}]_E = \begin{pmatrix} 5+1 \\ 5-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} \longrightarrow \text{or } [\vec{x}]_E = U[\vec{x}]_F \qquad (1)$$

where $U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Example to motivate change of Basis

Consider $\vec{X} = (6,4)$ or $(\vec{X})_E = (6,4)$

or $\vec{x} = 6\hat{e}_1 + 4\hat{e}_2 = 6(1,0) + 4(0,1)$

We will adopt now the column vector representation of the vectors.

$$(\hat{X})_E = 6\begin{pmatrix}1\\0\end{pmatrix} + 4\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}6\\4\end{pmatrix}$$

The vector $\begin{pmatrix}6\\4\end{pmatrix}$ is called the coordinates of $\vec{x}$ with respect to the standard basis $E = \{\hat{e}_1, \hat{e}_2\} = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$

Next, we want to find the coordinates of $\vec{x}$ with respecto to the basis $F = \{\hat{u}_1, \hat{u}_2\} = \{\overset{(\hat{u}_1)_E}{\begin{pmatrix}1\\1\end{pmatrix}}, \overset{(\hat{u}_2)_E}{\begin{pmatrix}1\\-1\end{pmatrix}}\}$

$\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ are the coordinates with respect to the standard basis for u_1 and u_2, respectively.

Question: Find the coordinates of $\hat{x}$ with respect to basis F.

Answer:

$$(\vec{x})_E = \begin{pmatrix}6\\4\end{pmatrix} = \overbrace{\alpha_1(\vec{u}_1)_E + \alpha_2(\vec{u}_2)_E}^{\text{we want to determine } \alpha_1 \text{ and } \alpha_2} = \alpha_1\begin{pmatrix}1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\-1\end{pmatrix} =$$

$$= \begin{pmatrix}\alpha_1+\alpha_2\\ \alpha_1-\alpha_2\end{pmatrix}$$

$$\Rightarrow \begin{cases}\alpha_1+\alpha_2=6\\ \alpha_1-\alpha_2=4\end{cases} \qquad \begin{bmatrix}1 & 1 & 6\\ 1 & -1 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 6\\ 0 & -2 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 6\\ 0 & -1 & -1\end{bmatrix}$$

$$\Rightarrow \alpha_2 = 1, \quad \alpha_1 = 6-1 = 5$$

$$\text{or} \quad (\vec{x})_F = \begin{pmatrix}5\\1\end{pmatrix}$$

Notice that

$$(\vec{x})_E = \begin{pmatrix}6\\4\end{pmatrix} = \overset{=5}{\alpha_1}\begin{pmatrix}1\\1\end{pmatrix} + \overset{=1}{\alpha_2}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{array}{cc} (\vec{u}_1)_E & (\vec{u}_2)_E \\ \downarrow & \downarrow \end{array} \!\!\! \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\overset{=5}{\alpha_1}\\ \overset{=1}{\alpha_2}\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}(\vec{x})_F$$

Then

$$\boxed{(\vec{x})_F = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}^{-1}(\vec{x})_E} \qquad (2.1)$$

The matrix $P = \begin{array}{cc} (\vec{u}_1)_E & (\vec{u}_2)_E \\ \downarrow & \downarrow \end{array}\!\!\!\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ is called the transition matrix from the basis F to the basis E (standard basis)

From (2.1)

$$(\vec{x})_F = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} (\vec{x})_E$$

Now, $$P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

therefore, $$(\vec{X})_F = \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \overset{\overset{\vec{x}}{\downarrow}}{\begin{pmatrix} 6 \\ 4 \end{pmatrix}}$$

or $$(\vec{x})_F = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} (\vec{x})_E .$$

In fact, $$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

As a consequence, the matrix

$$P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$ is the transition matrix

from the basis E (standard) to the new basis F.

Formula (1) is true for any vector $\vec{z} \in \mathbb{R}^2$
and an arbitrary basis $F = \{\vec{u}_1, \vec{u}_2\}$

In fact, if

$[\vec{z}]_F = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ equivalent to $\vec{z} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2$

$[\vec{u}_1]_E = \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix}$ " " $\vec{u}_1 = u_{11} \vec{e}_1 + u_{21} \vec{e}_2$

$[\vec{u}_2]_E = \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix}$ " " $\vec{u}_2 = u_{12} \vec{e}_1 + u_{22} \vec{e}_2$

Then, $[\vec{z}]_E = U [\vec{z}]_F$

In fact, $\vec{z} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 = \alpha_1 (u_{11} \vec{e}_1 + u_{21} \vec{e}_2) +$

$+ \alpha_2 (u_{12} \vec{e}_1 + u_{22} \vec{e}_2)$

or $\vec{z} = (\alpha_1 u_{11} + \alpha_2 u_{12}) \vec{e}_1 + (\alpha_1 u_{21} + \alpha_2 u_{22}) \vec{e}_2$

Which is equivalent to

$$[\vec{z}]_E = \begin{pmatrix} \alpha_1 u_{11} + \alpha_2 u_{12} \\ \alpha_1 u_{21} + \alpha_2 u_{22} \end{pmatrix} = \underset{\overset{\shortparallel}{U}}{\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$= U [\vec{z}]_F$$

or $\boxed{[\vec{z}]_E = U [\vec{z}]_F}$ (3.1)

Normally, we know the coordinates of $\vec{z}$ with respect to E and we want to determine the coordinates of $\vec{z}$ with respect to a new basis F. This is an easy task from (3.1).

In fact, the columns of U are linearly independent vectors. Therefore, U is invertible and

$$[\vec{z}]_F = U^{-1}[\vec{z}]_E$$

Changing basis from $B_1 = \{\vec{v}_1, \vec{v}_2\}$ to $B_2 = \{\vec{u}_1, \vec{u}_2\}$

Problem: Assume we know the coordinates of $\vec{z}$ with respect to $B_1 \overset{=\{\vec{v}_1,\vec{v}_2\}}{,}$ $[\vec{z}]_{B_1} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ and we want to determine the coordinates of $\vec{z}$ with respect to $B_2 \overset{=\{\vec{u}_1,\vec{u}_2\}}{,}$ $[\vec{z}]_{B_2} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$.

Let $[\vec{v}_1]_E = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}$, $[\vec{v}_2]_E = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$

$[\vec{u}_1]_E = \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix}$, $[\vec{u}_2]_E = \begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix}$

According to our previous result,

$$[\vec{z}]_E = V \underset{\text{Known}}{[\vec{z}]_{B_1}}, \quad \text{where } V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

Also

$$[\vec{z}]_E = U \underset{\text{Unknown}}{[\vec{z}]_{B_2}}, \quad \text{where } U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

Therefore,

$$V [\vec{z}]_{B_1} = U [\vec{z}]_{B_2}$$

and

$$\boxed{[\vec{z}]_{B_2} = U^{-1} V [\vec{z}]_{B_1}}$$

Consider the vector $\vec{x}$, whose coords. with respect to standard basis $E=\{\hat{e}_1, \hat{e}_2\}$ are $(\vec{x})_E = \begin{pmatrix}6\\4\end{pmatrix}$

Also, we have shown that coords. of $\vec{x}$ with respect to

$$F = \{\vec{u}_1, \vec{u}_2\} = \left\{ \overset{(\vec{u}_1)_E}{\begin{pmatrix}1\\1\end{pmatrix}}, \overset{(\vec{u}_2)_E}{\begin{pmatrix}1\\-1\end{pmatrix}} \right\}$$ are

$$(\vec{x})_F = \begin{pmatrix}5\\1\end{pmatrix}.$$

Find the coords. of $\vec{x}$ with respect to the basis

$$B = \left\{ \overset{(\vec{b}_1)_E}{\begin{pmatrix}2\\1\end{pmatrix}}, \overset{(\vec{b}_2)_E}{\begin{pmatrix}1\\3\end{pmatrix}} \right\},$$ Knowing that the coords. of $(\vec{x})_F = \begin{pmatrix}5\\1\end{pmatrix}$.

$$(\vec{x})_E = P(\vec{x})_F \quad \text{or} \quad \overset{(\vec{x})_E}{\begin{pmatrix}6\\4\end{pmatrix}} = \overset{(\vec{u}_1)_E\ (\vec{u}_2)_E}{\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}} \overset{(\vec{x})_F}{\begin{pmatrix}5\\1\end{pmatrix}}$$

Also

$$(\vec{x})_E = Q(\vec{x})_B \quad \text{or} \quad \overset{(\vec{x})_E}{\begin{pmatrix}6\\4\end{pmatrix}} = \overset{(\vec{b}_1)_E\ (\vec{b}_2)_E}{\begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}} \overset{(\vec{x})_B}{\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix}}$$

$$\Rightarrow \begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}\begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}5\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\beta_1\\\beta_2\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}5\\1\end{pmatrix}$$

$$\begin{pmatrix}2 & 1\\1 & 3\end{pmatrix}^{-1} = $$

$$= \frac{1}{5}\begin{pmatrix}3 & -1\\-1 & 2\end{pmatrix} =$$

$$= \begin{pmatrix}3/5 & -1/5\\-1/5 & 2/5\end{pmatrix}$$ ✓

or

$$(\vec{X})_B = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 18/5 - 4/5 \\ -6/5 + 8/5 \end{pmatrix} = \begin{pmatrix} 14/5 \\ 2/5 \end{pmatrix}.$$

or $(\vec{X})_B = \begin{pmatrix} 14/5 \\ 2/5 \end{pmatrix}$ and $(\vec{X})_F = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$$(\vec{x})_B = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 2/5 & 4/5 \\ 1/5 & -3/5 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = P_{F\to B} (x)_F$$

or $(\vec{x})_B = P_{F\to B} (\vec{x})_F$

$\downarrow$

"Transition matix from F to B."

Check that $P_{F\to B} = \left[(\vec{u}_1)_B , (\vec{u}_2)_B \right]$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2c_1 + c_2 \\ c_1 + 3c_2 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 + 3c_2 = 1 \\ 2c_1 + c_2 = 1 \\ \hline -5c_2 = -1 \Rightarrow c_2 = +1/5 \end{array} \Rightarrow (\vec{u}_1)_B = \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} \checkmark$$

$$\Rightarrow c_1 = 1 - 3/5 = 2/5$$