5.2 Diagonalization

Start finding eigenvalues and eigenvectors of

Exercise #17)

$$\boxed{A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}}$$ Show next page

Eigenvalues and Eigenvectors:

$\lambda_1 = 1 \longrightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ basis for eigenspace corresp. to $\lambda_1 = 1$.

$\lambda_2 = 2 \longrightarrow \left\{ \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \right\}$ basis for eigenspace corresp to $\lambda_2 = 2$

$\lambda_3 = 3 \longrightarrow \left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right\}$ basis for eigenspace corresp. to $\lambda_3 = 3$.

Define

$$P = \begin{array}{c} \begin{matrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix} \end{array}$$

The set of vectors $\{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ are lin. indep.

In fact, Solving $P\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \boxed{x_3 = 0} \Downarrow \boxed{x_2 = 0} \Downarrow \boxed{x_1 = 0}$$

Exercise #13 Section 7.2 — Find Eigenvalues and Eigenvectors. Determine whether A is diagonalizabl

$$A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}, \quad |\lambda I - A| = 0 \Leftrightarrow \begin{vmatrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{vmatrix} = 0$$

Characteristic polyn.: $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Equivalent to: $(\lambda-1)(\lambda-2)(\lambda-3) = 0$

Therefore, eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

Finding eigenvectors

$\lambda_1 = 1$ $(\lambda_1 I - A)\vec{x} = \vec{0}$ (Solns. of this equ.)

$$\begin{bmatrix} 2 & -4 & 2 & 0 \\ 3 & -3 & 0 & 0 \\ 3 & -1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_2 = x_3 \\ x_1 = 2x_2 - x_3 = \end{matrix}$$

Therefore, eigenvectors corresponding to $\lambda_1 = 1$

$$\vec{x} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\rightarrow$ Basis for eigenspace Corresponding to $\lambda_1 = 1$.

$\underline{\lambda_2 = 2}$ $\quad (\lambda_2 I - A)\vec{x} = \vec{0}$

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ 3 & -2 & 0 & 0 \\ 3 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_2 &= x_3 \\ 3x_1 &= 4x_2 - 2x_3 \\ &= 2x_3 \\ \text{or } x_1 &= 2/3\, x_3 \end{aligned}$$

Eigenvectors corresponding to $\lambda_2 = 2$.

$$\vec{x} = \begin{bmatrix} 2/3\, x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2/3 \\ 1 \\ 1 \end{bmatrix} \text{ or } x_3 \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

Basis for eigensp[ace] Corresp. to $\lambda_2 = 2$

$\underline{\lambda_3 = 3}$ $\quad (\lambda_3 I - A)\vec{x} = \vec{0}$.

$$\begin{bmatrix} 4 & -4 & 2 & 0 \\ 3 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1/2 & 0 \\ 0 & 2 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_2 &= \frac{3}{4} x_3 \\ x_1 &= \frac{1}{4} x_3 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 1/4\, x_3 \\ 3/4\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1/4 \\ 3/4 \\ 1 \end{bmatrix} \text{ or } x_3 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Basis for eigenspace Corresp. to $\lambda_3 = 3$

Defining $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Verify $P^{-1}AP = D$.

Since only solution of $P\vec{x}=\vec{0}$ is $\vec{x}=\vec{0}$ the set of eigenvectors $\{\vec{p}_1,\vec{p}_2,\vec{p}_3\}$ is linearly independent. and also the matrix P is invertible.

In fact,

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 3 & -2 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5 & 3 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

So $\boxed{P^{-1} = \begin{bmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}}$

Notice that

$$P^{-1}AP = \begin{bmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & -5 & 3 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 3 \\ 1 & 6 & 9 \\ 1 & 6 & 12 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D} \Rightarrow \boxed{P^{-1}AP = D}$$

Example 1.- (book) Section 7.2

$$A=\begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}, \quad |\lambda I - A| = 0 \Leftrightarrow \begin{vmatrix}\lambda & 0 & 2\\ -1 & \lambda-2 & -1\\ -1 & 0 & \lambda-3\end{vmatrix} = 0$$

Characteristic polyn.: $\lambda(\lambda-2)(\lambda-3)+2(\lambda-2)=0$

$\lambda^3-5\lambda^2+8\lambda-4=0$ or $(\lambda-1)(\lambda-2)^2=0 \Rightarrow \begin{cases}\lambda_1=1\\ \lambda_2=2\end{cases}$

Finding eigenvectors:

$\lambda_1=1$ $\quad (\lambda_1 I - A)\vec{x}=0$

$$\left[\begin{array}{ccc|c}1&0&2&0\\-1&-1&-1&0\\-1&0&-2&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&2&0\\0&-1&1&0\\0&0&0&0\end{array}\right] \Rightarrow \begin{array}{l}x_1=-2x_3\\ x_2=x_3\end{array}$$

$\Rightarrow$ Eigenvs. $\vec{x}=\begin{bmatrix}-2x_3\\x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}-2\\1\\1\end{bmatrix}$ → Basis for the eigenspace Corresp. to $\lambda_1=1$.

For $\lambda_2=2$

$$\left[\begin{array}{ccc|c}2&0&2&0\\-1&0&-1&0\\-1&0&-1&0\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&1&0\\0&0&0&0\\0&0&0&0\end{array}\right] \Rightarrow$$

$x_1=-x_3$, x_2, x_3 free

$$\vec{x}=\begin{bmatrix}-x_3\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$$

→ Basis for eigensp. $\lambda_2=2$

Therefore,

$$P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Verify $P^{-1}AP = D$.

<u>Example 2.</u>- (Sect. 7.2)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix} \Rightarrow |\lambda I - A| = 0 \Leftrightarrow \begin{vmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ 3 & -5 & \lambda-2 \end{vmatrix} = 0$$

Charact. polyn: $(\lambda-1)(\lambda-2)^2 = 0$

<u>$\lambda_1 = 1$</u> $(\lambda_1 I - A)\vec{x} = 0$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 3 & -5 & -1 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -8 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 = -x_2 = \frac{1}{8}x_3 \\ x_2 = \frac{-1}{8}x_3 \end{array}$$

$$\vec{x} = x_3 \begin{bmatrix} 1/8 \\ -1/8 \\ 1 \end{bmatrix} \text{ or } x_3 \begin{bmatrix} 1 \\ -1 \\ 8 \end{bmatrix}$$

$\lambda_2 = 2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 3 & -5 & 0 & 0 \end{array}\right] \sim \begin{bmatrix} 3 & -5 & 0 & 0 \\ 0 & 5/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_2 = 0 \\ 3x_1 = 5x_2 \Rightarrow x_1 = 0 \\ x_3 \text{ free.} \end{array}$$

$$\vec{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ Eigensp. basis.

$$\boxed{\text{There is not } P \text{ s.t. } PAP^{-1} = D}$$

Def: - A, B $n \times n$ matrix

<u>B is similar to A</u> if there is an invertible matrix P ($n \times n$) such that

$$\boxed{P^{-1} A P = B}$$

Notice that if B is similar to A, then A is similar to B since

$$A = P B P^{-1}$$

Renaming $P^{-1} = Q \Rightarrow \boxed{A = Q^{-1} B Q}$

<u>Importance</u>: Similar matrices have properties in common.

For example,

$$\det(B) = \det(P^{-1} A P) =$$

$$= \det(P^{-1}) \det(A) \det(P) =$$

$$= \frac{1}{\cancel{\det}(P)} \cdot \det(A) \cancel{\det}(P) = \det(A)$$

We will say that $\underline{\det}$ is <u>invariant under similarity</u>

See <u>Table 1</u> for other similarity invariant properties.

Same eigenvalues
rank
nullity, etc

If the matrix B of the previous definition happens to be diagonal, then the matrix A is a special one.

Def:- A $(n \times n)$ matrix is said to be diagonalizable if it's similar to some diagonal matrix D. It means there is P $(n \times n)$ invertible such that

$$\boxed{P^{-1}AP = D}$$

Ask students to work on

$AP = ?$ $\qquad$ $PD = ?$

where $P = [\vec{p}_1\ \vec{p}_2\ \cdots\ \vec{p}_n]$, $\quad D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & O & & \ddots & \lambda_n \end{bmatrix}$

$$AP = [A\vec{p}_1\ A\vec{p}_2\ \cdots\ A\vec{p}_n]$$

$$PD = [\lambda_1\vec{p}_1\ \lambda_2\vec{p}_2\ \cdots\ \lambda_n\vec{p}_n]$$

$$AP = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & & p_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}p_{11} + \cdots + a_{1n}p_{n1} & \cdots \\ a_{21}p_{11} + \cdots + a_{2n}p_{n1} & \cdots \\ \vdots & \\ a_{n1}p_{11} + \cdots + a_{nn}p_{n1} & \cdots \end{bmatrix} = [A\vec{p}_1 \cdots]$$

Then, if $\quad AP = PD \Rightarrow A\vec{p}_1 = \lambda_1\vec{p}_1, \ldots, A\vec{p}_n = \lambda_n\vec{p}_n$

This means what?

It means that $\vec{p}_1, \ldots, \vec{p}_n$ are eigenvectors of A if they are nonzero.

Thm 5.2.1 $\quad A$ $n\times n$ matrix

A diagonalizable $\Leftrightarrow$ A has n linearly indep. eigenvectors.

Proof.- ($\rightarrow$) If A is diagonalizable

then $\quad P^{-1}AP = D \Rightarrow \underline{AP = PD}$

From previous Computation if $D = \begin{bmatrix} \lambda_1 & 0 \cdots & 0 \\ 0 & \ddots & \lambda_n \end{bmatrix}$

if $\quad P = [\vec{p}_1 \cdots \vec{p}_n]$ then

$$A\vec{p}_1 = \lambda_1 \vec{p}_1, \ \ldots, \quad A\vec{p}_n = \lambda_n \vec{p}_n$$

Also, from hypothesis $\underline{P \text{ is invertible}} \Rightarrow \underline{\{\vec{p}_1, \ldots, \vec{p}_n\}}$

$\underline{\text{is lin. indep.}} \Rightarrow \vec{p}_i \neq 0, \quad i = 1, 2 \ldots n$

Therefore, $\underline{\text{All } \vec{p}_i \ (i=1,..,n) \text{ are eigenvectors of } A}$

$\underline{\text{Corresponding to the eigenvalues } \lambda_i \ (i=1,..,n)}$.

($\leftarrow$) If A has n lin. indep. eigenvectors $\{\vec{p}_1, .., \vec{p}_n\}$

with corresp. eigenvalues $\lambda_1, \lambda_2, .., \lambda_n$ (they may be repeated)

then, for $P = [\vec{p}_1 \ \vec{p}_2 \cdots \vec{p}_n]$

$$AP = [A\vec{p}_1 \ \cdots \ A\vec{p}_n] = [\lambda_1\vec{p}_1 \ \ \lambda_2\vec{p}_2 \ \cdots \ \lambda_n\vec{p}_n]$$

$$= [\vec{p}_1 \ \vec{p}_2 \cdots \vec{p}_n] \begin{bmatrix} \lambda_1 & & 0 \\ 0 & \ddots & \lambda_n \end{bmatrix}$$

$$= \mathbf{PD}$$

This matrix P is invertible because its columns are lin. indep.

or $\quad AP = PD \Rightarrow P^{-1}AP = D$

$\Rightarrow \boxed{A \text{ is diagonalizable}}$

For diagonalizing procedure review previous problems.

Thm 5.5.2 — $\{\vec{v}_1, \vec{v}_2, .., \vec{v}_k\}$ Set of eigenvectors of A

— $\lambda_1, \lambda_2, .., \lambda_k$ Corresponding eigenvalues

— and all λ_i's are differents.

$\Rightarrow \{\vec{v}_1, .., \vec{v}_k\}$ are lin. independent.

Proof.— (by induction)

If $k=1$ $S = \{\vec{v}_1\}$ is lin. indep. because $\vec{v}_1 \neq \vec{0}$.

otherwise, $\vec{v}_1$ wouldn't be an eigenvector.

Assume, it's true for $k = h$

It means a set of eigenvectors $S = \{\vec{v}_1, .., \vec{v}_h\}$ corresponding to distinct eigenvalues is lin. indep.

We want to prove now, that the thm. is true for $k = h+1$

Consider $$C_1 \vec{v}_1 + C_2 \vec{v}_2 + \cdots + C_h \vec{v}_h + C_{h+1} \vec{v}_{h+1} \quad (10.1)$$

$\Rightarrow$ Multiplying by A

$$C_1 A\vec{v}_1 + C_2 A\vec{v}_2 + \cdots + C_h A\vec{v}_h + C_{h+1} A\vec{v}_{h+1} = \vec{0}$$

or $$C_1 \lambda_1 \vec{v}_1 + C_2 \lambda_2 \vec{v}_2 + \cdots + C_h \lambda_h \vec{v}_h + C_{h+1} \lambda_{h+1} \vec{v}_{h+1} = \vec{0} \quad (10.2)$$

Multiplying (10.1) by λ_{h+1} and subtrating (10.2)

$$c_1\lambda_{h+1}\vec{v}_1 + c_2\lambda_{h+1}\vec{v}_2 + \cdots + c_h\lambda_{h+1}\vec{v}_h + c_{h+1}\lambda_{h+1}\vec{v}_{h+1} = \vec{0}$$

$$- \quad c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_h\lambda_h\vec{v}_h + c_{h+1}\lambda_{h+1}\vec{v}_{h+1} = \vec{0}$$

$$\Rightarrow \quad c_1(\overset{\neq 0}{\lambda_{h+1}-\lambda_1})\vec{v}_1 + c_2(\overset{\neq 0}{\lambda_{h+1}-\lambda_2})\vec{v}_2 + \cdots + c_h(\overset{\neq 0}{\lambda_{h+1}-\lambda_h})\vec{v}_h + c_{h+1}(\cancel{\lambda_{h+1}-\lambda_{h+1}})^{0}\vec{v}_{h+1} = \vec{0}$$

Using that λ_i are different and $\{\vec{v}_1, .., \vec{v}_h\}$ is lin. indep.

$$\Longrightarrow \quad \boxed{c_1 = 0, \ \ldots \ c_h = 0}$$

Subst. into (10.1)

$c_{h+1}\vec{v}_{h+1} = \vec{0}$; but $\vec{v}_{h+1} \neq 0$ (eigenvector)

$$\Rightarrow \boxed{c_{h+1} = 0}$$

As a result, all coefficients $c_i = 0, \quad i = 1, .., h, h+1.$

and the set $S = \{\vec{v}_1, .., \vec{v}_h, \vec{v}_{h+1}\}$ is lin. indep.

Thm 5.2.3 A $(n\times n)$

- A has n distinct eigenvalues

then, A is diagonalizable.

Proof:- If A has n distinct eigenvalues $\overset{\text{Thm 5.2.2}}{\Rightarrow}$ the corresponding eigenvectors $\{\vec{p}_1, \dots, \vec{p}_n\}$ form a lin. indep. set of vectors.

then by theorem 5.2.1 A is diagonalizable

Powers of a diagonalizable matrix

If A is diagonalizable

$$P^{-1}AP = D \Rightarrow A = PDP^{-1}$$

$$\Rightarrow \quad A^2 = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

In general, $A^k = PD^kP^{-1}$.

and $$D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

Find out A^k for matrix A of Ex. #17.

Geometric and Algebraic Multiplicity

In exercise (17), we found that

$$\mathcal{P}(\lambda) = (\lambda-1)(\lambda-2)(\lambda-3)$$

So $\lambda_1 = 1 \longrightarrow \vec{V}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\lambda_2 = 2 \longrightarrow \vec{V}_2 = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$, $\lambda_3 = 3 \longrightarrow \vec{V}_3 = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

In each case, the <u>multiplicity</u> of the root λ_i in $\mathcal{P}(\lambda)$ equals the dimension of the corresponding eigenspace, which is always "one".

In <u>example 1</u> (A diagonalizable)

$$\mathcal{P}(\lambda) = (\lambda-1)(\lambda-2)^2$$

$\lambda_1 = 1 \rightarrow \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\lambda_2 = 2 \rightarrow \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

corresponding Eigenspace dimension = 1 dimension = 2

So, again algebraic multiplicity agrees with dimension of corresponding eigenspace.

In <u>example 2</u> (A not diagonalizable)

$$\mathcal{P}(\lambda) = (\lambda-1)(\lambda-2)^2$$

$\lambda_1 = 1 \rightarrow \begin{pmatrix} 1 \\ -1 \\ 8 \end{pmatrix}$, $\lambda_2 = 2 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

So, "algebraic multiplicity" of $\lambda_2 = 2$ is "2" but dimension of corresp. eigenspace is "1".

Def. — If λ_0 is an eigenvalue of a matrix A then the algebraic multiplicity of λ_0 is the number of times that $\lambda - \lambda_0$ appears as a factor in the expression for $P(\lambda)$.

On the other hand, the dimension of the eigenspace corresponding to λ_0 is called the "geometric multiplicity" of λ_0".

Back to previous examples.

Thm 5.2.5 A $(n \times n)$

a) For every eigenvalue A

$$\boxed{\text{Geom. multipl.} \leq \text{alg. multiplicity}}$$

b) A is diagonalizable iff for each eigenvalue λ

$$\boxed{\text{Geom multipl. of } \lambda = \text{alg. multipl. of } \lambda.}$$