Do

Thm 1.2.2 Sect. 1.2 pp. 19

Consider

$$\begin{cases} 3x_1 + 6x_2 + 7x_3 = 0 \\ -x_1 - 2x_2 + 2x_3 = 0 \end{cases}$$

2 equations
3 unknowns

what do we know about this?

$$\begin{bmatrix} 3 & 6 & 7 & 0 \\ -1 & -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & 2 & 0 \\ 3 & 6 & 7 & 0 \end{bmatrix} -$$

$$\sim \begin{bmatrix} 1 & +2 & -2 & 0 \\ 3 & 6 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & +2 & -2 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

only two unknown can be determined one at least will be free.

$$13x_3 = 0 \Rightarrow \boxed{x_3 = 0}$$

$$\boxed{x_1 = -2x_2}$$

Inf. many solns.

Same for any system with n unkns and $m < n$ equations. If $n = m + r$, then only m unknowns can be determined at most and the system will have $n - r$ free variables at least.

4.5 Dimension

Thm 4.5.2 i) V finite-dimensional vector space

ii) $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis

Then,

a) Any set with more than n vectors is lin. indep.

b) Any set with less than n vectors does not span V.

Proof:-

a) Let $\tilde{S} = \{\vec{u}_1, \dots, \vec{u}_m\}$ $m > n$. we want to prove $\tilde{S}$ is lin. dep.

First, each u_i, $i = 1, \dots, m$ can be expressed as lin. comb. of $\vec{v}_s$

In fact,

$$\begin{cases} u_1 = a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + a_{31}\vec{v}_3 + \dots + a_{n1}\vec{v}_n \\ u_2 = a_{12}\vec{v}_1 + a_{22}\vec{v}_2 + \dots \qquad + a_{n2}\vec{v}_n \\ \vdots \\ u_m = a_{1m}\vec{v}_1 + a_{2m}\vec{v}_2 + \dots \qquad + a_{nm}\vec{v}_n \end{cases}$$

Now, we will consider a lin. comb. of U_i's $= \vec{0}$ and try to prove that all coeffs. are not zero.

$$K_1 \vec{u}_1 + \cdots + K_m \vec{u}_m = \vec{0}$$

$$K_1 (a_{11} \vec{v}_1 + \cdots + a_{n1} \vec{v}_n) + \cdots + K_m (a_{1m} \vec{v}_1 + \cdots + a_{nm} \vec{v}_m) = \vec{0}$$

$$\Rightarrow (K_1 a_{11} + \cdots + K_m a_{1m}) \vec{v}_1 + \cdots + (K_1 a_{n1} + \cdots + K_m a_{nm}) \vec{v}_m = \vec{0}$$

But $\vec{v}$'s are l. indep., then

$$\begin{cases} a_{11} K_1 + \cdots + a_{1m} K_m = 0 \\ \vdots \\ a_{n1} K_1 + \cdots + a_{nm} K_m = 0 \end{cases}$$

m unknws $>$ n equs.

$\Rightarrow$ inf. many solns.

In particular nontrivial solns.

At least one $K_j \neq 0$. $j = 1, .., m$

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b) Let $\hat{S} = \{ \vec{w}_1, .., \vec{w}_r \}$, $r < n$ $7q \Rightarrow 7p$ type-proof

If $\hat{S}$ span V in particular

$$\begin{cases} \vec{v}_1 = a_{11} \vec{w}_1 + \cdots + a_{r1} \vec{w}_r \\ \vdots \\ \vec{v}_n = a_{1n} \vec{w}_1 + \cdots + a_{rn} \vec{w}_r \end{cases}$$

Now, consider: $K_1 \vec{v}_1 + \cdots + K_n \vec{v}_n = \vec{0}$ Want to show not all Ks are zero.

$$k_1 \vec{v}_1 + \cdots + k_n \vec{v}_n = \vec{0}$$

$$\Rightarrow \quad k_1 (a_{11} \vec{w}_1 + \cdots + a_{r1} \vec{w}_r) + \cdots + k_n (a_{1n} \vec{w}_1 + \cdots + a_{rn} \vec{w}_r) = \vec{0}$$

$$\Rightarrow \quad (k_1 a_{11} + \cdots + k_n a_{1n}) \vec{w}_1 + \cdots + (k_1 a_{r1} + \cdots + k_n a_{rn}) \vec{w}_r = \vec{0}$$

$$\Rightarrow \quad \begin{cases} k_1 a_{11} + \cdots + k_n a_{1n} = 0 \\ \vdots \\ k_1 a_{r1} + \cdots + k_n a_{rn} = 0 \end{cases}$$

r equs $< n$ unknws

Inf. many solns

So, $k_j \neq 0$. for some j.

<u>thm 4.5.</u> All bases for finite-dimensional vector space have same number of vectors.

<u>Proof.</u> Let

and $B = \{\vec{v}_1, \ldots, \vec{v}_m\}$ a basis for V

According to 4.5.2 (a) $m \leq n$ $\Rightarrow$ $\underline{\underline{m = n}}$

" " 4.5.2 (b) $m \geq n$

Def: The dimension of any finite-dimensional vector space V ($\dim(V)$) is the number of vectors in any basis of V.

The set $\{\vec{0}\}$ has dimension $= 0$.

Thm 4.5.3 (Plus/minus theorem)

- $S = \{\vec{v}_1, \vec{v}_2, .., \vec{v}_r\}$.

a) If S is lin. indep. and $\vec{v} \notin \text{Span}(S)$ then $S \cup \{\vec{v}\}$ is lin. indep.

b) If $\vec{v}_j \in S$ is such that $\vec{v}_j \in \text{Span}\{\vec{v}_1, .., \vec{v}_{j-1}, \vec{v}_{j+1}, .., \vec{v}_r\}$ then $S - \{\vec{v}_j\}$ is such that
$$\text{Span}(S) = \text{Span}(S - \{\vec{v}_j\}).$$

Proof.- a) Consider

$$k_1\vec{v}_1 + \cdots + k_r\vec{v}_r + k_{r+1}\vec{v} = \vec{0}$$

want to prove $k_1 = 0 = \cdots = k_{r+1}$

Two possibilities for k_{r+1}

i) $k_{r+1} = 0 \Rightarrow k_1\vec{v}_1 + \cdots + k_r\vec{v}_r = \vec{0} \xRightarrow{\text{Hyp. l. indep}} k_i = 0,\ i = 1, .., r$ done

ii) $k_{r+1} \neq 0 \Rightarrow \vec{v} = \frac{k_1}{-k_{r+1}}\vec{v}_1 + \cdots + \frac{k_r}{-k_{r+1}}\vec{v}_r$

$\Rightarrow \vec{v} \in \text{Span}(S)$ Contradiction!

then $k_{r+1} = 0$.

b) If $V_j \in \text{span}\left(\{\vec{V}_1,.., \vec{V}_{j-1}, \vec{V}_{j+1},.., \vec{V}_r\}\right) = \text{Span}\left(S-\{\vec{V}_j\}\right)$

$\Rightarrow \quad V_j = K_1\vec{V}_1 + \cdots + K_{j-1}\vec{V}_{j-1} + K_{j+1}\vec{V}_{j+1} + \cdots + K_r\vec{V}_r$

Now, consider $\vec{v} \in \text{Span}\left(\{\vec{V}_1,.., \vec{V}_j,.., \vec{V}_r\}\right) = \text{Span}(S)$

then, $\vec{v} = C_1\vec{V}_1 + \cdots + C_j\vec{V}_j + \cdots + C_r\vec{V}_r$

$= C_1\vec{V}_1 + \cdots + C_j\left(K_1\vec{V}_1 + \cdots + K_{j-1}\vec{V}_{j-1} + K_{j+1}\vec{V}_{j+1} + \cdots + K_r\vec{V}_r\right)$

$+ \cdots C_r\vec{V}_r =$

$= (C_1 + C_jK_1)\vec{V}_1 + \cdots + (C_{j-1} + C_jK_{j-1})\vec{V}_{j-1} + (C_{j+1} + C_jK_{j+1})\vec{V}_{j+1} +$

$+ \cdots + C_r\vec{V}_r = \vec{0}$

$\Rightarrow \quad \vec{v} \in \text{Span}\{\vec{V}_1,.., \vec{V}_{j-1}, \vec{V}_{j+1},.., \vec{V}_r\} = \text{Span}\left(S-\{\vec{V}_j\}\right)$

$\Rightarrow \quad \text{Span}\left(\{\vec{V}_1,.., \vec{V}_r\}\right) = \text{Span}(S) = \text{Span}\left(S-\{\vec{V}_j\}\right).$

Thm 4.5.4 – V n-dimensional vector space
– $S = \{\vec{v}_1, .., \vec{v}_n\}$

Then, S is a basis for V if and only if
(a) S spans V (Span $(V) = S$) or
(b) S is linearly independent.

Proof.- ($\rightarrow$) trivial

($\leftarrow$) If Span $(V) = S$, then there are two
(a) possibilities for S: i) S is lin. indep $\Rightarrow$ S is basis
done

or ii) S is lin. dep. $\Rightarrow$ $V = \text{Span}\{S - \{v_j\}\}$
but this contradicts thm 4.5.2 (b).
Less than n vectors does not span V.

(b) It's similar using thm 4.5.2. (a).

Thm 4.5.5 – S finite set of V (Vector space finite-dim.)
a) Span $(S) = V$, but not a basis $\Rightarrow$ S can be reduced to a basis.

b) S is lin. indep, but not a basis $\Rightarrow$
S can be enlarged to a basis.

Proof.- Easy based on thm. 4.5.3