4.4 A basis for $\mathbb{R}^3$

#3

$$B = \{ (3,1,-4), (2,5,6), (1,4,8) \}$$

Lin. indep.

i) $C_1 (3,1,-4) + C_2 (2,5,6) + C_3 (1,4,8) = \vec{0}$

$$\begin{cases} 3C_1 + 2C_2 + C_3 = 0 \\ C_1 + 5C_2 + 4C_3 = 0 \\ -4C_1 + 6C_2 + 8C_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 1 & 5 & 4 & 0 \\ -4 & 6 & 8 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 5 & 4 & 0 \\ 3 & 2 & 1 & 0 \\ -4 & 6 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 4 & 0 \\ 0 & -13 & -11 & 0 \\ 0 & 26 & 24 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 5 & 4 & 0 \\ 0 & -13 & -11 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \Rightarrow$$

Backward Subst.

$$2C_3 = 0 \Rightarrow \boxed{C_3 = 0}$$

$$-13C_2 - 11\overset{=0}{C_3} = 0 \Rightarrow -13C_2 = 0 \Rightarrow \boxed{C_2 = 0}$$

$$C_1 + 5\overset{=0}{C_2} + 4\overset{=0}{C_3} = 0 \Rightarrow \boxed{C_1 = 0}$$

Then, $B = \{ (3,1,-4), (2,5,6), (1,4,8) \}$ is a linearly independent set.

ii) $\text{Span}(B) = \mathbb{R}^3$.

Given an arbitrary $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, we want to determine if there are $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1(3,1,-4) + c_2(2,5,6) + c_3(1,4,8) = \vec{b} = (b_1, b_2, b_3)$$

$$\begin{cases} 3c_1 + 2c_2 + c_3 = b_1 \\ c_1 + 5c_2 + 4c_3 = b_2 \\ -4c_1 + 6c_2 + 8c_3 = b_3 \end{cases}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & b_1 \\ 1 & 5 & 4 & b_2 \\ -4 & 6 & 8 & b_3 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 5 & 4 & b_2 \\ 3 & 2 & 1 & b_1 \\ -4 & 6 & 8 & b_3 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & 4 & b_2 \\ 0 & -13 & -11 & b_1 - 3b_2 \\ 0 & 26 & 24 & b_3 + 4b_2 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 5 & 4 & b_2 \\ 0 & -13 & -11 & b_1 - 3b_2 \\ 0 & 0 & 2 & \underbrace{b_3 + 4b_2 + 2b_1 - 6b_2}_{b_3 - 2b_2 + 2b_1} \end{array}\right]$$

then,

$$\boxed{c_3 = \frac{b_3 - 2b_2 + 2b_1}{2}},$$

$$\boxed{c_2 = \frac{b_1 - 3b_2 + 11c_3}{-13}}, \quad \boxed{c_1 = b_2 - 4c_3 - 5c_2}$$

Coordinates relative to basis B

If $\vec{b} = (1, 2, 3)$

then $c_3 = \dfrac{3-4+2}{2} = \dfrac{1}{2}$,

$c_2 = \dfrac{1-6+\frac{11}{2}}{-13} = \dfrac{\frac{1}{2}}{-13} = -\dfrac{1}{26}$

$c_1 = 2 - 4\left(\frac{1}{2}\right) - 5\left(-\frac{1}{26}\right) = \cancel{2} - \cancel{2} + \frac{5}{26} = \frac{5}{26}$

then, $\boxed{(1,2,3) = \frac{5}{26}(3,1,-4) - \frac{1}{26}(2,5,6) + \frac{1}{2}(3,1,-4).}$

$(1,2,3)_B = \left(\frac{5}{26}, -\frac{1}{26}, \frac{1}{2}\right)$

Examples of Subspaces and basis

1) Matrices 2×2 $\quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad$ Vector space M_{22}

usual operations for matrices.

Consider $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

This set of matrices Spans M_{22}

Why?

Are they linearly independent?

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$$ $c_1 = 0 \ldots, c_4 = 0$ ✓

So, linearly indep.

How about the two

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Span $(\{M_1, M_4\})$ = All diagonal matrices 2×2

Are they linearly independent?

Thms. Sect 5.3

Thm 5.3.1 S with 2 or more vectors

a) S is lin. dep. if one vector can be written as lin. comb. of the others.

b) S is lin. indep. if and only if no vector is lin. comb. of the others.

Thm 5.3.3

$S = \{\vec{v}_1, \ldots, \vec{v}_r\}$ in $\mathbb{R}^n$. If $r > n \Rightarrow S$ is lin dep.

Thm 5.3.2

a) Any set S containing $\vec{0}$ is lin. dep.

b) A set $S = \{\vec{v}_1, \vec{v}_2\}$ is lin. indep. if and only if $\vec{v}_1 \neq k\vec{v}_2$, for any k.

Proof. Thm 5.3.2.

a) Any set $\{\vec{v}_1, v_2, .., \vec{v}_n, \vec{0}\}$ is lin. dep.

Since $$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n + 3\vec{0} = \vec{0}$$

If $c_1 = c_2 = \cdots = c_n = 0$ but the coeff. for $\vec{0}$ is ~~≠0~~ not zero.

b) $S' = \{\vec{v}_1, \vec{v}_2\}$

Def. l. indep $\Leftrightarrow$ $[k_1\vec{v}_1 + k_2\vec{v}_2 = \vec{0} \Rightarrow k_1 = 0,\ k_2 = 0.]$

($\longrightarrow$) l. indep $\Rightarrow$ neither vector is scalar multiple of the other.

$p \Rightarrow q$ equiv. $\neg q \Rightarrow \neg p$

$\neg q$: So if $\vec{v}_1 = k\vec{v}_2$, $k \neq 0$.

$\Rightarrow$ $\vec{v}_1 - k\vec{v}_2 = \vec{0}$ and coeffs. are not zero

therefore $\{\vec{v}_1, \vec{v}_2\}$ is not l. indep. $\rightarrow \neg p$.

($\longleftarrow$) Neither $\vec{v}_1$ or $\vec{v}_2$ is multiple of the other $\Rightarrow$ $\{\vec{v}_1, \vec{v}_2\}$ is lin. indep.

$p \Rightarrow q$.

$\neg q \Rightarrow \neg p$ is $\{\vec{v}_1, \vec{v}_2\}$ lin. dep $\Rightarrow$ $\vec{v}_1$ or $\vec{v}_2$ is multiple of the other.

If $\neg q$ then exist k_1 and k_2 not both zeros s.t.

$k_1\vec{v}_1 + k_2\vec{v}_2 = \vec{0}$, if $k_1 \neq 0 \Rightarrow \vec{v}_1 = -\frac{k_2}{k_1}v_2$ ($\neg p$) ✓

14 $\{\vec{v}_1, \vec{v}_2\}$ l. indep and $\vec{v}_3$ not in span $\{\vec{v}_1, \vec{v}_2\}$

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is lin. indep.

Proof.— $p \Rightarrow q$ We will prove $\neg q \Rightarrow \neg p$.

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ were lin. dependent but $\{\vec{v}_1, \vec{v}_2\}$ are lin. indep.

Then there is a combination

$$K_1\vec{v}_1 + K_2\vec{v}_2 + K_3\vec{v}_3 = 0 \quad \text{and } K_3 \neq 0$$

If not, $K_3 = 0$ and l. indep of $\{\vec{v}_1, \vec{v}_2\} \Rightarrow K_1 = 0, K_2 = 0$

and therefore $\{\vec{v}_1, v_2, \vec{v}_3\}$ were l. indep Contrary to our assumption

So $K_3 \neq 0 \Rightarrow \boxed{\vec{v}_3 = -\frac{K_1}{K_3}\vec{v}_1 - \frac{K_2}{K_3}\vec{v}_2}$.

$\vec{v}_3$ is in span $\{\vec{v}_1, \vec{v}_2\}$.

19 c $\{1, \sin x, \sin 2x\}$

$$K_1(1) + K_2 \sin x + K_3 \sin 2x = O(x), \quad \text{for all } x.$$

this should be an identity for every x

Compare with $\{1, x^2, 2x^2 - 4\}$

In particular

i) $x = 0 \Rightarrow K_1(1) = 0 \Rightarrow K_1 = 0$.

ii) $x = \frac{\pi}{2} \Rightarrow K_2 \underbrace{\sin\frac{\pi}{2}}_{=1} + K_3 \underbrace{\sin(\pi)}_{0} = 0 \Rightarrow K_2 = 0$.

iii) $x = \pi/4 \Rightarrow K_3 \underbrace{\sin\frac{\pi}{2}}_{=1} = 0 \Rightarrow K_3 = 0$.

Exercises 5.3

(4) (a) $\{\overset{P_1}{1+3x+3x^2}, \overset{P_2}{x+4x^2}, \overset{P_3}{5+6x+3x^2}, \overset{P_4}{7+2x-x^2}\}$

P_2 : set of polynomials of degree ≤ 2.

$$k_1 P_1 + k_2 P_2 + k_3 P_3 + k_4 P_4 = 0 \text{ polyn} = 0 + 0x + 0x^2$$

therefore

$1: \; k_1 + 0 + 5k_3 + 7k_4 = 0$

$x: \; 3k_1 + k_2 + 6k_3 + 2k_4 = 0$

$x^2: \; 3k_1 + 4k_2 + 3k_3 - k_4 = 0$

lin. dep. since # unknowns > # of eqns.

Using thm 1.2.1 the above System has nontrivial solution

It means any of $k's \neq 0$.

(10) $S = \{\vec{v}_1, .., \vec{v}_r\}$ lin. indep. $\Rightarrow$ Any subset $W \subset S$ is also l. indep.

Proof.– Let $\{\vec{v}_1, \vec{v}_2, .., \vec{v}_m\} \subset S$: $m \leq r$.

and $\vec{v}_{m+1}, .., \vec{v}_r$ not in S

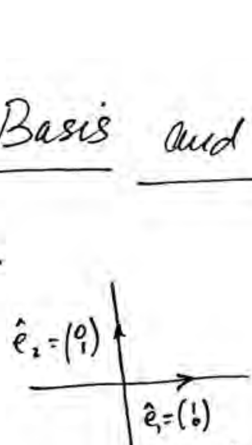
then If $k_1 \vec{v}_1 + \cdots + k_m \vec{v}_m = \vec{0} \Rightarrow$

$$k_1 \vec{v}_1 + \cdots + k_m \vec{v}_m + 0\vec{v}_{m+1} + \cdots + 0\vec{v}_r = \vec{0} \overset{\text{hyp}}{\Rightarrow} k_1 = k_2 = \cdots = k_m = 0.$$

Remark: If the vectors in the subset were not the first m the proof is completely analogous.

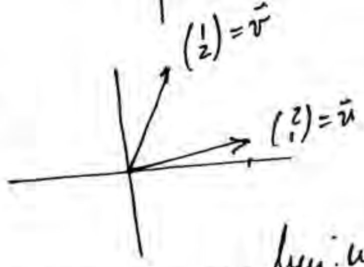
5.4 Basis and dimension

Consider.



$\{\hat{e}_1, \hat{e}_2\}$ standard basis ?

Also



What do they have in Common ?

These sets are lin. indep in $\mathbb{R}^2$ and $\text{Span}\{\binom{1}{0}, \binom{0}{1}\} = \mathbb{R}^2$, $\text{Span}\{\binom{2}{1}, \binom{1}{2}\} = \mathbb{R}^2$

they will be called basis for $\mathbb{R}^2$.

Def.– V Vect. Space, $S = \{\vec{v}_1, .., \vec{v}_n\}$ in V

If a) S is lin. indep., b) S spans V

S will be called a basis for V.

<u>Example 1.</u>– Consider the two basis in $\mathbb{R}^2$

$B_1 = \{\hat{e}_1, \hat{e}_2\}$ and $B_2 \{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \}$

Write the vector $\vec{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ as a liner comb. of vectors

in B_1 and B_2.

$$\begin{pmatrix} +4 \\ -1 \end{pmatrix} = (+4) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Also, $\begin{pmatrix} 4 \\ -1 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$

$(\vec{V})_{B_1} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, $(\vec{V})_{B_2} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. This vector is called the coord. vector of $\vec{V}$ relative to B_2.

<u>Example 5.</u>– Standard basis for $\mathcal{P}_n$.

$\{1, x, x^2, \ldots, x^n\}$ — lin. indep, span $\mathcal{P}_n$. Verify!

<u>Example 6.</u>– Standard basis for $M_{2\times 2}$

$M_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Verify it!

CHAPTER 5 (Cont)

Thm 5.4.1 (Uniqueness of Basis repr.)

$B = \{\vec{v}_1, \vec{v}_2, .., \vec{v}_n\}$ a basis in V, then any $\vec{v} \in V$ can be written as

$$\vec{v} = c_1 \vec{v}_1 + \cdot + c_n \vec{v}_n \quad \text{in exactly one way.}$$

Proof.– Assume there is a second way to represent $\vec{v}$ as a linear comb. of vectors in B

$$\vec{v} = k_1 \vec{v}_1 + \cdot + k_n \vec{v}_n$$

$$\Rightarrow \quad \vec{0} = (c_1 - k_1)\vec{v}_1 + \cdot + (c_n - k_n)\vec{v}_n \xRightarrow{\text{l. indep cond.}}$$

$$c_1 = k_1, \ldots, c_n = k_n$$

Def.– The scalars $c_1, c_2 ., c_n$ are called the coords. of $\vec{v}$ relative to the basis B.

notation $\boxed{(\vec{v})_B = (c_1, c_2, ., c_n) \text{ is called coord. vector of } \vec{v} \text{ relative to } B.}$

Example for matrices:

$\frac{17}{24}$
$\overline{41}$

$$3\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + 1\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + (-3)\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 41 \\ 40 & 2 \end{bmatrix}.$$ (Transparency)

Def.— V is called finite-dimensional if it contains a finite set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ that forms a basis. If not V is infinite-dimensional.

$\{\vec{0}\}$ V. space is finite-dimensional

Examples: $\mathbb{R}^n$, P_n, M_{mn} F-D

$F(-\infty,\infty)$, $C(-\infty,\infty)$ real valued conts fns. : inf-dimensional

<u>Example.</u>- Coordinate vector of $M = \begin{bmatrix} 13 & 41 \\ 40 & 2 \end{bmatrix}$ relative to the base (exerc. 5)

$$B = \left\{ \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \right\}$$

Observe that

$$3\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + 1\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} - 3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 41 \\ 40 & 2 \end{bmatrix}$$

Then the vector

$$(M)_B = \begin{pmatrix} 3 \\ 1 \\ -3 \\ 4 \end{pmatrix}$$ is coord. vector of M relative to B.

How can you obtain this coord. vector?