4.9 Matrix Transformation
8.1 General Linear Transformation

① Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 & 5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}$$

For $\vec{x} \in \mathbb{R}^4$
the matrix product : $A\vec{x} \in \mathbb{R}^3$

Therefore, a function T can be defined as

$$T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$
$$\vec{x} \longrightarrow A\vec{x}$$

This funcion T receives the name of transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.

② Transformations are actually more general.
Consider the function
$$T: P_n \to P_{n+1}$$

$$p = p(x) = c_0 + c_1 x + \cdots + c_n x^n \longrightarrow T(p) = T(p(x)) = x\,p(x)$$

or $T(p) = T(p(x)) = x\left(c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n\right) =$
$$= c_0 x + c_1 x^2 + c_2 x^3 + \cdots + c_n x^{n+1}.$$

Def.- For vector spaces V and W if T is a function that assigns to a vector $\vec{v} \in V$ an image vector $\vec{w} \in W$, T is called a transformation from V to W. If $V = W$ T is called an operator on V.

$$T: V \longrightarrow W$$
$$\vec{v} \longrightarrow \vec{w}$$

The matrix transformation T of example ① can also be written as

$$T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$
$$\vec{x} = (x_1, x_4) \longrightarrow \vec{w} = (w_1, w_2, w_3)$$

$$\boxed{\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 - 5x_4 \\ w_2 &= 4x_1 + x_2 - 2x_3 + x_4 \\ w_3 &= 5x_1 - x_2 + 4x_3 \end{aligned}}$$

The matrix form of T is given by

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The matrix $A = \begin{pmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{pmatrix}$ receives the name of standard matrix for the transformation T.

Also, when a transformation T is defined from a matrix A, as indicated above, T is denoted as T_A.

Properties of T_A:

Given $T_A: \mathbb{R}^n \to \mathbb{R}^m$ defined from a matrix $A_{m\times n}$, for all vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $k \in \mathbb{R}$

a) $T_A(\vec{0}) = \vec{0}$, since $A\vec{0} = \vec{0}$

b) $T_A(k\vec{u}) = k\,T_A(\vec{u})$. (Homogeneity property) $\to$ since $A(k\vec{u}) = kA\vec{u}$

c) $T_A(\vec{u}+\vec{v}) = T_A(\vec{u}) + T_A(\vec{v})$ (Additivity property) $\to$ Since $A(\vec{u}+\vec{v}) = A\vec{u}+A\vec{v}$.

As a consequence of (b) and (c)

$$\boxed{T_A(k_1\vec{u}_1 + k_2\vec{u}_2 + \cdots + k_r\vec{u}_r) = k_1 T_A(\vec{u}_1) + k_2 T_A(\vec{u}_2) + \cdots + k_r T_A(\vec{u}_r)}$$

Ex. #6) Consider $T: \mathbb{R}^2 \to \mathbb{R}^2$
$(x,y) \to (w_1, w_2)$

b) $T(x,y) = (-y, x)$

$$\left.\begin{array}{l} w_1 = -y \\ w_2 = x \end{array}\right\} \Rightarrow \text{matrix form} \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

d) $T(x,y) = (x^2, y)$

$w_1 = x^2$
$w_2 = y$

There is not a matrix form that leads to a x^2 term.

Notice in example ① that

$$A\vec{e}_1 = A\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}2 & -3 & 1 & -5\\4 & 1 & -2 & 1\\5 & -1 & 4 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}2\\4\\5\end{pmatrix}$$

$$A\vec{e}_2 = \begin{pmatrix}-3\\1\\-1\end{pmatrix}, \quad A\vec{e}_3 = A\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = 0\begin{pmatrix}2\\4\\5\end{pmatrix} + 0\begin{pmatrix}-3\\1\\-1\end{pmatrix} + 1\begin{pmatrix}1\\-2\\4\end{pmatrix} + 0\begin{pmatrix}-5\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\-2\\4\end{pmatrix}$$

and $A\vec{e}_4 = \begin{pmatrix}-5\\1\\0\end{pmatrix} \Rightarrow A = [A\vec{e}_1 \ A\vec{e}_2 \ A\vec{e}_3 \ A\vec{e}_4]$

In general given a matrix A $m \times n$.

$$A = [\vec{c}_1 \ \vec{c}_2 \ \dots \ \vec{c}_n]$$

$$A\vec{e}_1 = \vec{c}_1, \dots, \quad A\vec{e}_n = \vec{c}_n$$

Then, if the transf. T_A is defined as : $T_A : \mathbb{R}^n \to \mathbb{R}^m$

$\boxed{\vec{w} = T_A \vec{x}}$ Math formula.

to find A (standard matrix of T_A), it's enough to find the images of the basis vectors : $A\vec{e}_i \quad i = 1, 2, \dots, n$

and define $A = [A\vec{e}_1 \ A\vec{e}_2 \dots A\vec{e}_n]$.

#9) Find standard matrix for the operator

$$T: \mathbb{R}^3 \to \mathbb{R}^3$$

$$(x_1, x_2, x_3) \to (w_1, w_2, w_3)$$

$$w_1 = 3x_1 + 5x_2 - x_3$$
$$w_2 = 4x_1 - x_2 + x_3$$
$$w_3 = 3x_1 + 2x_2 - x_3$$

$$T(\vec{e}_1) = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}, \quad T(\vec{e}_2) = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}, \quad T(\hat{e}_3) = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{pmatrix}$$

#12) d)

$$T: \mathbb{R}^2 \to \mathbb{R}^3$$

$$(x_1, x_2) \to (w_1, w_2, w_3)$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

or

$$w_1 = -x_1 + x_2$$
$$w_2 = 2x_1 + 4x_2$$
$$w_3 = 7x_1 + 8x_2$$

General linear Transformation

Consider two vector spaces V and W.
T is called a linear transformation from V to W

$$T: V \longrightarrow W.$$

If for all $\vec{u}, \vec{v} \in V$ and $k \in \mathbb{R}$

i) $T(k\vec{u}) = k\,T(\vec{u})$ (Homogeneity prop.)

ii) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ (Additivity prop.)

If $V = W$, T is called a linear operator on V

Thm 8.1.1 If $T: V \to W$ is a linear transformation, then:

i) $T(\vec{0}) = \vec{0}$

ii) $T(k_1\vec{v}_1 + k_2\vec{v}_2) = k_1 T(\vec{v}_1) + k_2 T(\vec{v}_2)$.

Example 4 (book)

$$T: V \longrightarrow W$$
$$\vec{x} \to T(\vec{x}) = k\vec{x}$$

It's linear since

i) $T(c\vec{u}) = k(c\vec{u}) = c(k\vec{u}) = c\,T(\vec{x})$.

ii) $T(\vec{u}+\vec{v}) = k(\vec{u}+\vec{v}) = k\vec{u} + k\vec{v} = T(\vec{u}) + T(\vec{v})$.

Example 5 (boon)

$$T: P_n \to P_{n+1}$$
$$\vec{p} \to T(\vec{p}) = T(p(x)) = x\,p(x)$$

Linear

i) $T(k\vec{p}) = T(k\,p(x)) = x\,(k\,p(x)) = k\,(x\,p(x)) = kT(p)$

ii) $T(\vec{p}_1 + \vec{p}_2) = T((p_1+p_2)(x)) = x\,(p_1+p_2)(x) = x\,p_1(x) + x\,p_2(x)$

$= T(\vec{p}_1) + T(\vec{p}_2)$

Sect. 8.1

Exercise #7 $\quad T: P_2 \to P_2$

a) $a_0 + a_1 x + a_2 x^2 \longrightarrow a_0 + a_1(x+1) + a_2(x+1)^2$

b) $a_0 + a_1 x + a_2 x^2 \longrightarrow (a_0+1) + (a_1+1)x + (a_2+1)x^2$

Are they linear Transformation?

(b) is not. Because

$$T(k\vec{p}) = T(ka_0 + ka_1 x + ka_2 x^2) = (ka_0+1) + (ka_1+1)x + (ka_2+1)x^2$$

$$\neq k(a_0+1) + k(a_1+1)x + k(a_2+1)x^2 = kT(\vec{p}).$$

Def. $T: V \to W$ linear transformation

a) Kernel of $T = Ker(T) = \{ \vec{v} \in V \mid T(\vec{v}) = \vec{0} \}$.

b) range of $T = R(T) = \{ \vec{w} \in W \mid$ there is $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w} \}$

It might be more than one $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

For $T = T_A : \mathbb{R}^n \to \mathbb{R}^m$

$Ker(T_A) =$ null space of A
$R(T_A) = Col(A)$

In Example 5 (book)

$T(\vec{p}) = T(p(x)) = x\,p(x)$

$Ker(T) = \{\vec{0}\}$, $\quad R(T) = \{ a_0 x + a_1 x^2 + \dots + a_n x^{n+1} \}$
Subspace

Thm 8.1.3 $\quad T: V \to W$ linear transf.

a) $Ker(T)$ is a subsp.

b) $R(T)$ is a subsp.

Proof.- $\vec{0} \in Ker(T)$ because $T(\vec{0}) = \vec{0}$.

Also If $\vec{v}, \vec{w} \in Ker T \Rightarrow T(k_1\vec{v} + k_2\vec{w}) = k_1 \underbrace{T(\vec{v})}_{=\vec{0}} + k_2 \underbrace{T(\vec{w})}_{=\vec{0}} =$
$= \vec{0} + \vec{0} = \vec{0}$.

(b) $\vec{0} \in R(T)$, Since $T(\vec{0}) = \vec{0}$. and $\vec{0} \in V$.

If $\vec{w}_1$ and $\vec{w}_2$ are in W

Then, there exists $\vec{v}_1, \vec{v}_2 \in V$ Such that

$T(\vec{v}_1) = \vec{w}_1$ and $T(\vec{v}_2) = \vec{w}_2$

$k_1\vec{w}_1 + k_2\vec{w}_2$ is also in $R(T)$

Since $T(\underbrace{k_1\vec{v}_1 + k_2\vec{v}_2}_{\in V \text{ because } V \text{ is vector space}}) \overset{\text{linearity of } T}{=} k_1 T(\vec{v}_1) + k_2 T(\vec{v}_2)$

$= k_1\vec{w}_1 + k_2\vec{w}_2$.

<u>Def</u> $T: \underset{\text{vector space}}{V} \to \underset{\text{vector space}}{W}$ linear transf.

If $R(T)$ is finite dimensional, then

rank of T = rank(T) = dim $(R(T))$

If $\ker(T)$ is finite dimensional, then

nullity of T = nullity(T) = dim $(\ker(T))$.

<u>Thm 8.1.4</u> $T: \underset{\text{finite dim v.s.}}{V} \to \underset{\text{v.s.}}{W}$ dim $(V) = n$

$$\boxed{\text{rank}(T) + \text{nullity}(T) = n}$$

Proof. For $1 \le \dim(\ker(T)) < n$

or $\dim(\ker(T)) = r$ $\quad 1 \le r < n$

Consider a basis $B = \{\vec{v}_1, .., \vec{v}_r\} \subset V$

Then B can be extended to a basis

$$\hat{B} = \{\underbrace{\vec{v}_1, .., \vec{v}_r}, \underbrace{\vec{v}_{r+1}, .., \vec{v}_n}\} \text{ of } V$$

If we can prove that $S = \{T\vec{v}_{r+1}, .., T\vec{v}_n\}$ is a basis for $R(T)$ then the thm would be proved.

First, S spans $R(T)$, since if $\vec{b} \in R(T)$, then there is $\vec{v} \in V$ that

$T(\vec{v}) = \vec{b}$, but $\vec{v} = c_1\vec{v}_1 + .. + c_r\vec{v}_r + c_{r+1}\vec{v}_{r+1} + .. + c_n\vec{v}_n$

$$\Rightarrow \vec{b} = T(\vec{v}) = c_1 \overset{=0}{T(\vec{v}_1)} + .. + c_r \overset{=0}{T(\vec{v}_r)} + c_{r+1}T(\vec{v}_{r+1}) + .. + c_n T(\vec{v}_n)$$

$$= c_{r+1}T(\vec{v}_{r+1}) + .. + c_n T(\vec{v}_n) \in \text{Span}(S)$$

Secondly S is linearly indep.

Consider $k_{r+1}T(\vec{v}_{r+1}) + k_{r+2}T(\vec{v}_{r+2}) + .. + k_n T(\vec{v}_n) = \vec{0}$

$\Rightarrow T(k_{r+1}\vec{v}_{r+1} + .. + k_n\vec{v}_n) = \vec{0} \Rightarrow k_{r+1}\vec{v}_{r+1} + .. + k_n\vec{v}_n \in \ker(T)$

$\Rightarrow k_{r+1}\vec{v}_{r+1} + .. + k_n\vec{v}_n = k_1\vec{v}_1 + .. + k_r\vec{v}_r$ $\quad \{\vec{v}_1, .., \vec{v}_r\}$ is a basis $\ker(T)$.

$\Rightarrow k_1\vec{v}_1 + k_2\vec{v}_2 + .. + k_r\vec{v}_r - k_{r+1}\vec{v}_{r+1} - .. - k_n\vec{v}_n = \vec{0}$

$\xRightarrow[\text{of } \{\vec{v}_1, .., \vec{v}_n\}]{\text{Lin. indep.}} k_1 = 0, .., k_r = 0, k_{r+1} = 0, .., k_n = 0$ ✓