

4.8 Rank, Nullity and the Fund. Matrix Spaces.

In our previous examples, we found that
 for A and R (row equivalent matrices)
 the following relationship holds:

- i) A basis for $\text{row}(A)$ is a basis for $\text{row}(R)$
 and viceversa.
- ii) A basis for $\text{col}(A)$ is not a basis for $\text{col}(R)$
 and viceversa (in general)
- iii) A basis for $\text{col}(A)$ and a basis for $\text{col}(R)$
 always have the same number of vectors.

Therefore,

$$\dim(\text{row}(A)) = \dim(\text{row}(R)) \quad (1.1)$$

$$\dim(\text{col}(A)) = \dim(\text{col}(R)) \quad (1.2)$$

Thm 4.8.1 $\dim(\text{row}(A)) = \dim(\text{col}(A)) \quad (1.3)$

Proof

It will be sufficient to show that
 $\dim(\text{row}(R)) = \dim(\text{col}(R))$
 and then use (1.1) and (1.2).

This is true because

$$\dim(\text{row}(R)) = \# \text{ of nonzero rows} \Rightarrow$$

$$\dim(\text{Col}(R)) = \# \text{ of leading 1's} //$$

or rows with a leading "1".
These are the same.

Definition

- the common dimension of $\text{row}(A)$ and $\text{Col}(A)$ is

called rank of A or $\text{rank}(A)$.

- Also $\dim(\text{null}(A))$ is called nullity of A and is denoted by $\text{nullity}(A)$.

For the matrix

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}_{\substack{4 \times 5 \\ m \times n}}$$

we found that

$$\text{rank}(A) = 3$$

$$\text{nullity}(A) = 2$$

thus, $\text{rank}(A) + \text{nullity}(A) = 5 = n = \# \text{ of columns.}$

This result is true in general.

Remark: For a matrix A $m \times n$

All vectors in $\text{row}(A)$ are in $\mathbb{R}^n \Rightarrow \dim(\text{row}(A)) \leq n$

All vectors in $\text{col}(A)$ are in $\mathbb{R}^m \Rightarrow \dim(\text{col}(A)) \leq m$

but $\dim(\text{row}(A)) = \dim(\text{col}(A)) = \text{rank}(A)$

then, $\text{rank}(A) \leq \min(m, n)$.

Theorem 4.8.2

If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

Proof

Given $A\vec{x} = \vec{0}$, A $m \times n$

This system has n variables. These variables are of

two types

$$\boxed{\text{Leading Variables}} + \boxed{\text{Free Variables}} = n$$

$$\parallel \qquad \qquad \parallel$$

$$\text{rank}(A) + \text{nullity}(A) = n$$

Overdetermined and Underdetermined Systems

Over-determined: $m > n$

$$a) \begin{cases} x_1 - x_2 = 1 \\ 3x_1 + x_2 = 2 \\ -x_1 + 2x_2 = 3 \end{cases} \quad \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 3 & 1 & 2 \\ -1 & 2 & 3 \end{array} \right] \sim$$

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \\ -1 & 2 \end{bmatrix}_{3 \times 2} \quad m > n$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 4 & -1 \\ 0 & 1 & 4 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -1/4 \\ 0 & 0 & 13/4 \end{array} \right]$$

$0 = 13/4$ inconsistent

No solution

b) $\begin{cases} x_1 - x_2 = 0 \\ 3x_1 + x_2 = 4 \\ -x_1 + 2x_2 = 1 \end{cases} \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & 1 & 4 \\ -1 & 2 & 1 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 4 & 4 \\ 0 & 1 & 1 \end{array} \right] \sim$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{cases} x_2 = 1 \\ x_1 = 1 \end{cases} \text{ Consistent!}$$

$\text{rank}(A) = 2 = r$

Unique soln., not free variables. Also, $\text{nullity}(A) = n - \text{rank}(A) = 2 - 2 = 0$

$$\text{Col}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\} \subset \mathbb{R}^3$$

Obviously, there are vectors \vec{b} in \mathbb{R}^3 such that

$\vec{b} \notin \text{col}(A)$ or $A\vec{x} = \vec{b}$ does not have a soln.

b) Notice

$$(1) \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + (1) \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \in \text{Col}(A).$$

Also,

$$\text{if } \begin{cases} x_1 - x_2 = 0 & \text{Eq. 1} \\ 3x_1 + x_2 = 4 & \text{Eq. 2} \\ -x_1 + 2x_2 = 1 & \text{Eq. 3} \end{cases}$$

Then

$$\begin{aligned} -\frac{7}{4} \times (\text{Eq. 1}) &\Rightarrow -\frac{7}{4}x_1 + \frac{7}{4}x_2 = 0 \\ + \frac{1}{4} \times (\text{Eq. 2}) &\Rightarrow \frac{3}{4}x_1 + \frac{1}{4}x_2 = 1 \\ \hline &\Rightarrow -x_1 + 2x_2 = 1 \quad \text{Same as eq. 3} \end{aligned}$$

But

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \notin \text{Col}(A).$$

there are not x_1 and x_2 such that

$$x_1 \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$c) \begin{cases} x_1 - x_2 = 1 \\ 3x_1 - 3x_2 = 3 \\ -x_1 + x_2 = -1 \end{cases} \quad \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 3 & -3 & 3 \\ -1 & 1 & -1 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] \Rightarrow \begin{aligned} x_1 &= 1 + x_2 \\ x_2 &\text{ is a free variable.} \end{aligned}$$

Inf. many solus.

Underdetermined

$$\underline{m < n}$$

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 + x_2 - x_3 = 2 \end{cases}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\text{Augm. matrix} = \begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 2 & 1 & -1 & | & 2 \end{bmatrix} \sim$$

$$= \begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 0 & 3 & -3 & | & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{x_2 = x_3}, \quad x_1 = 1 + x_2 - x_3, \quad \boxed{x_3 \text{ free variable}}$$

$$\boxed{x_1 = 1}$$

Any soln. can be written as

$$\vec{x} = \begin{pmatrix} 1 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_3 \\ x_3 \end{pmatrix} =$$

$$\text{or } \vec{x} = x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{x} \in \text{Null}(A) \text{ iff } \vec{x} = x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \text{nullity}(A) = 1$$

Also, $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\text{Col}(A)$

$$\Rightarrow \text{rank}(A) = 2.$$

What can be said about?

$$A\vec{x} = \vec{b} \quad (1) \quad \text{When}$$

$$A\vec{x} = \vec{0} \quad (2) \quad A_{m \times n}$$

for arbitrary
 m and n .

Thm ~~5.6.5~~ Equiv. stats.

a) $A\vec{x} = \vec{b}$ is consistent.

b) \vec{b} is in the column space of A .

c) Coef matrix A and augmented $[A|\vec{b}]$ Same rank.

Thm ~~5.6.6~~ Equiv. stats.

a) $A\vec{x} = \vec{b}$ is consistent for every vector \vec{b} .

b) Column vectors of A span \mathbb{R}^m
i.e., $\text{Col}(A) = \mathbb{R}^m$.

c) $\text{rank}(A) = m$.

Some examples

$A_{m \times n}$

$m < n$

(I) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow A\vec{x} = \vec{0}$ has inf. many solns.
 $x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$A\vec{x} = \vec{b}$ Also has inf. many solns. for any \vec{b} since $Col(A) = \mathbb{R}^3$.

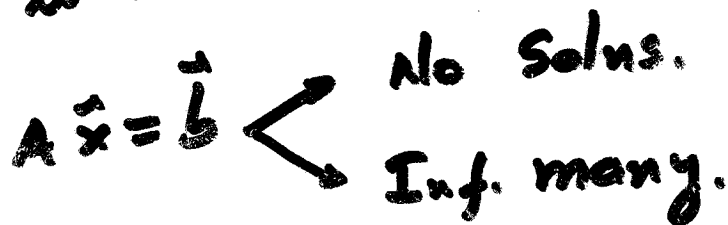
(II) $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow A\vec{x} = \vec{0}$ has inf. many solns.

but (a) $A\vec{x} = \vec{b}$ has no solution if \vec{b} is not in $Col(A)$.

(b) $A\vec{x} = \vec{b}$ has inf. many solns. if \vec{b} is in $Col(A)$.

Conclusion:

$A\vec{x} = \vec{0}$ has inf. many solns.



m > n

(I) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A\vec{x} = \vec{0}$ only triv. Soln.

$A\vec{x} = \vec{b}$ $\begin{cases} \text{No soln. } \vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ \text{Only one soln. } \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \end{cases}$

(II)

$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A\vec{x} = \vec{0}$, Inf. many Solns
Since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ lin. dep.

And

$A\vec{x} = \vec{b}$ $\begin{cases} \text{Inf. many solns. if } \vec{b} \text{ is in col}(A) \\ \vec{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \text{No soln. for a } \vec{b} \text{ not in col } A. \end{cases}$

Example 1.

Solns. of homogeneous⁴
System $A\vec{x} = \vec{0}$

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix} \text{max}$$

4x6

$$\sim \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$$

$$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_3 + 28x_4 + 37x_5 - 13x_6 \\ 2x_3 + 12x_4 + 16x_5 - 5x_6 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_3 \\ 2x_3 \\ x_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 28x_4 \\ 12x_4 \\ 0 \\ x_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 37x_5 \\ 16x_5 \\ 0 \\ 0 \\ x_5 \\ 0 \end{bmatrix} + \begin{bmatrix} -13x_6 \\ -5x_6 \\ 0 \\ 0 \\ 0 \\ x_6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Relationship Free Variables \leftrightarrow Nullity. $\left| \begin{array}{l} \text{Rank}(A) + \text{Nul}(A) \\ = 2 + 4 = 6 = n \end{array} \right.$

underdetermined. Thus, $A\mathbf{x} = \mathbf{b}$ must be consistent for some \mathbf{b} , and for each such \mathbf{b} the general solution must have $7 - r$ parameters, where r is the rank of A .

SUMMARY In Theorem 4.3.4 we listed eight results that are equivalent to the invertibility of a matrix A . We conclude this section by adding eight more results to the list to produce the following theorem that relates all of the major topics we have studied thus far.

Theorem 5.6.9 Equivalent Statements

If A is an $n \times n$ matrix, and if $T_A: R^n \rightarrow R^n$ is multiplication by A , then the following are equivalent.

- (a) A is invertible.
- (b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- (c) The reduced row-echelon form of A is I_n .
- (d) A is expressible as a product of elementary matrices.
- (e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix \mathbf{b} .
- (f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix \mathbf{b} .
- (g) $\det(A) \neq 0$.
- (h) The range of T_A is R^n .
- (i) T_A is one-to-one.
- (j) The column vectors of A are linearly independent.
- (k) The row vectors of A are linearly independent.
- (l) The column vectors of A span R^n .
- (m) The row vectors of A span R^n .
- (n) The column vectors of A form a basis for R^n .
- (o) The row vectors of A form a basis for R^n .
- (p) A has rank n .
- (q) A has nullity 0.

Proof. We already know from Theorem 4.3.4 that statements (a) through (i) are equivalent. To complete the proof we will show that (j) through (q) are equivalent to (b) by proving the sequence of implications $(b) \Rightarrow (j) \Rightarrow (k) \Rightarrow (l) \Rightarrow (m) \Rightarrow (n) \Rightarrow (o) \Rightarrow (p) \Rightarrow (q) \Rightarrow (b)$.

$(b) \Rightarrow (j)$. If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then by Theorem 5.6.8 the column vectors of A are linearly independent.

$(j) \Rightarrow (k) \Rightarrow (l) \Rightarrow (m) \Rightarrow (n) \Rightarrow (o)$. This follows from Theorem 5.4.5 and the fact that R^n is an n -dimensional vector space. (The details are omitted.)

Thm 4.8.5

- A $m \times n$ matrix
- $A\vec{x} = \vec{b}$ consistent
- A has rank r , $\text{rank}(A) = r$

then the system has $n-r$ free variables.

Proof.- From thm 4.8.2

$$\text{nullity}(A) = n - \text{rank}(A) = n - r$$

And $\text{nullity}(A) = \# \text{ Free Variables.} \Rightarrow$ ^{Using thm 4.7.2}

Thm 4.8.6 A $m \times n$ matrix

a) If $m > n$, then $A\vec{x} = \vec{b}$ is inconsistent for at least one vector $\vec{b} \in \mathbb{R}^m$. Typo in book

b) If $m < n$, then
(underdetermined)

$A\vec{x} = \vec{b}$ $\begin{cases} \rightarrow \text{It is inconsistent} \\ \text{or} \\ \rightarrow \text{It has inf. many solutions.} \end{cases}$

Another possibility for $m < n$ is

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 - 2x_2 + 2x_3 = 1 \end{cases}$$

$$\text{or } x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\text{or } (x_1 - x_2 + x_3) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 2 & -2 & 2 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 0 & 0 & 0 & | & -1 \end{bmatrix}$$

$0 = -1$
inconsistent

Inconsistent!

Summarizing

1) $A\vec{x} = \vec{b}$, A $m \times n$ and $m > n$

- It is inconsistent for some \vec{b} .

- The other two cases: 1) unique solution
2) Inf. many solns.
may appear

2) $A\vec{x} = \vec{b}$, A $m \times n$ and $m < n$

- It might be inconsistent when $\vec{b} \notin \text{Col}(A)$
Column vectors are lin. dep. and a basis for \mathbb{R}^m cannot be obtained from this l. dep. set of vectors.

- It might be consistent. In this case, there are inf. many solns., because there is at least one free variable.

Given A Consider A^T

$$\text{row}(A) = \text{Col}(A^T), \quad \text{Col}(A) = \text{row}(A^T)$$

Thm 4.8.7 $\text{rank}(A) = \text{rank}(A^T)$.

Proof. - Easy.

Def. - $W \subset \mathbb{R}^n$ Subspace
 $W^\perp \equiv \{ \vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0, \text{ for all } \vec{w} \in W \}$
 is called orthogonal complement.

Thm 4.8.8 $W \subset \mathbb{R}^n$ Subspace

- a) W^\perp is a Subspace of \mathbb{R}^n .
 - b) $W \cap W^\perp = \{ \vec{0} \}$
 - c) $(W^\perp)^\perp = W$.
-

Thm 4.8.9 A $m \times n$

- a) $(\text{null}(A))^\perp = \text{row}(A)$ in \mathbb{R}^n .
 - b) $(\text{null}(A^T))^\perp = \underset{\substack{\parallel \\ \text{row}(A^T)}}}{\text{Col}(A)}$ in \mathbb{R}^m
-