DEFINITION 1 Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

Vector space scalars can be real numbers or complex numbers. Vector spaces with real scalars are called ***real vector spaces*** and those with complex scalars are called ***complex vector spaces***. For now we will be concerned exclusively with real vector spaces. We will consider complex vector spaces later.

Observe that the definition of a vector space does not specify the nature of the vectors or the operations. Any kind of object can be a vector, and the operations of addition and scalar multiplication need not have any relationship to those on R^n. The only requirement is that the ten vector space axioms be satisfied. In the examples that follow we will use four basic steps to show that a set with two operations is a vector space.

To Show that a Set with Two Operations is a Vector Space

Step 1. Identify the set V of objects that will become vectors.

Step 2. Identify the addition and scalar multiplication operations on V.

Step 3. Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V. Axiom 1 is called ***closure under addition***, and Axiom 6 is called ***closure under scalar multiplication***.

Step 4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

Historical Note The notion of an "abstract vector space" evolved over many years and had many contributors. The idea crystallized with the work of the German mathematician H. G. Grassmann, who published a paper in 1862 in which he considered abstract systems of unspecified elements on which he defined formal operations of addition and scalar mul-

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We leave it as an

Example 1. $S = \{\vec{0}\}$, $\vec{0} + \vec{0} \overset{def}{=} \vec{0}$, $k\vec{0} \overset{def}{=} \vec{0}$

Example 2.- Matrices $M_{2\times 2}$

$$\vec{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$$

Def.- $\vec{u} + \vec{v} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{2}+v_{22} \end{bmatrix}$ is in $M_{2\times 2}$

$k\vec{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$ is in $M_{2\times 2}$

Easy to Verify all axioms : $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

All entries are real numbers and the real #s satisfy all the axioms. $-\vec{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$

Example 3.- Matrices $M_{m\times n}$

Example 4.- $F(-\infty, \infty) = \{f : \mathbb{R} \to \mathbb{R}\}$

$(f+g)(x) = f(x) + g(x)$ is in $F(-\infty,\infty)$

$(kf)(x) = kf(x)$. is in $F(-\infty,\infty)$

Ask for length $\|f\| = \,?$ Since values are real #s is also easy to prove all the other properties

Other possibilities : $F[a,b]$ or $F(a,b)$.

Example 5. – Not a vector space.

$V = \mathbb{R}^2$ but operations defined

Turn for similar exercise

a) $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$

b) $k\vec{u} = (k u_1, 0)$

omit

Everything is satisfied except which one?

$$1\vec{u} = 1(u_1, u_2) = (1u_1, 0) = (u_1, 0) \neq \vec{u}.$$

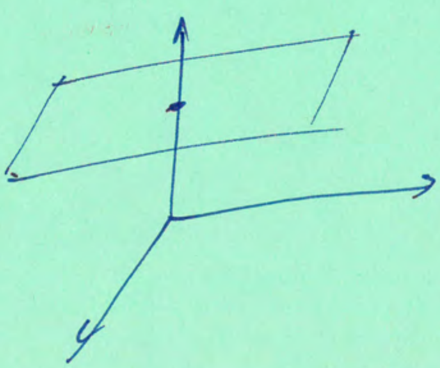
Example 6. – Every plane through the origin a vector space.

$$\mathcal{P} = \{ \vec{x} = (x, y, z) : \quad ax + by + cz = 0 \}$$ under usual operations.

Example 6'. – Not a vector space any plane $\mathcal{P}$ not passing through the origin.

all $\vec{x}$ satisfying

$$ax + by + cz = d \neq 0$$



$(0,0,0)$ is not in $\mathcal{P}$.

also $\hat{x} = (x_1, x_2, x_3)$ in $\mathcal{P}$

$\hat{y} = (y_1, y_2, y_3)$ in $\mathcal{P}$.

$\hat{x} + \hat{y}$ is not in $\mathcal{P}$.

In fact, $a(x_1 + y_1) + b(x_2 + y_2) + c(x_3 + y_3) = (ax_1 + \dots + cx_3) + (ay_1 + + cy_3) =$
$= d + d = 2d \neq d$

Example 8 (book) INTERESTING VECTOR SPACE

$V = \mathbb{R}$

Operations:

1) $u + v \overset{def}{=} uv$

2) $ku = u^k$

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then $1+1 = 1.1 = 1$ and $2(1) = 1^2 = 1$

All axioms are verified!

i) $\vec{0}$ vector is 1 or $\vec{0} = 1$.

ii) for u, $-u = \frac{1}{u}$

because $u + (-u) = u\frac{1}{u} = 1 = \vec{0}$.

iii) $k(u+v) = (uv)^k = u^k v^k = (ku)+(kv)$

iv) $u + (v+w) = (u+v)+w$

Since $u + (v+w) = u(v+w) = u(vw) = (uv)w =$

$= (u+v)w = (u+v) + w$

Thm 5.1.1 V is a vector space, $\vec{u}$ and $\vec{v}$ are in V and k is scalar, then

a) $0\vec{u} = 0$, b) $k\vec{0} = \vec{0}$, c) $(-1)\vec{u} = -\vec{u}$.

d) If $k\vec{u} = \vec{0}$, then $k = 0$ or $\vec{u} = \vec{0}$.

Proof.–

b) $k\vec{0} = \vec{0}$

$$k\vec{0} \overset{④}{=} k(\vec{0}+\vec{0}) \overset{⑦}{=} k\vec{0} + k\vec{0} \qquad (*)$$

Now, $-k\vec{0}$ exists (axiom 5). Adding it to both sides of equation (*).

$$\underbrace{k\vec{0} + (-k\vec{0})}_{Ⓐ} = \underbrace{(k\vec{0} + k\vec{0}) + (-k\vec{0})}_{Ⓑ}$$

Now, $Ⓐ = k\vec{0} + (-k\vec{0}) \overset{⑤}{=} \vec{0}$

and $Ⓑ \overset{③}{=} k\vec{0} + (k\vec{0} + (-k\vec{0})) \overset{⑤}{=} k\vec{0} + \vec{0} \overset{④}{=} k\vec{0}$.

Therefore, $Ⓐ = Ⓑ \Leftrightarrow \boxed{k\vec{0} = \vec{0}}$ ✓

(Good to emphasize logic)

d) If $k\vec{u}=\vec{0}$, then $k=0$ or $\vec{u}=\vec{0}$.

Proof.-

Assume $k\vec{u}=\vec{0}$, (1)

then there are two possibilities for k:

a) $k=0$ and the theorem is proved.

or

b) $k\neq 0$, then $\frac{1}{k}$ is a scalar too

and multiplying both sides of (1) by it

$$\underbrace{\frac{1}{k}(k\vec{u})}_{A} = \underbrace{\frac{1}{k}\vec{0}}_{B}$$

Now, $A \overset{(9)}{=} \left(\frac{1}{k}k\right)\vec{u} = 1\vec{u} \overset{(10)}{=} \vec{u}$

and $B = \frac{1}{k}\vec{0} \overset{(b)}{=} \vec{0}$

Therefore, $A=B \Leftrightarrow \boxed{\vec{u}=\vec{0}.}$

Unigness Theorem.

Thm.- Given a vector $\vec{u}$ in a vector space V, the negative vector $-\vec{u}$ is unique.

Proof.- If there were another vector $\vec{v}$ in V such that

$$\vec{u}+\vec{v}=\vec{0}, \qquad (1)$$

then adding $-\vec{u}$ to both sides of (1)

$$\underbrace{-\vec{u}+(\vec{u}+\vec{v})}_{=\,A} = \underbrace{-\vec{u}+\vec{0}}_{=\,B}$$

Now,

$$A = -\vec{u}+(\vec{u}+\vec{v}) \overset{(3)}{=} (-\vec{u}+\vec{u})+\vec{v} \overset{(5)}{=} \vec{0}+\vec{v} \overset{(4)}{=} \vec{v}$$

and

$$B = -\vec{u}+\vec{0} \overset{(4)}{=} -\vec{u}$$

Therefore $\quad A = B \Rightarrow \boxed{\vec{v}=-\vec{u}}$

And as a consequence $-\vec{u}$ is unique.

$S = \{moon, Sun\}$ Two elements set

It can't be a Vector Space.

Proof – One of the two element should be the $\vec{o}$ vector.

If $\vec{O} = Sun$

then we can define addition as

a) $moon + Sun = moon$

b) $Sun + Sun = Sun$

c) $moon + moon = Sun$ (moon needs to have a negative)

This is the only possibility for $\vec{O} = Sun$.

Scalar product definition

d) $K\,Sun = Sun$, for any K

e) $K\,moon = moon$, if $K \neq 0$. and $0\,moon = Sun$

Properties for the addition (1), (2), (3), (4), and (5) are Satisfied.

Property (7) $K\,(Sun + moon) = K\,moon = moon$, $K \neq 0$

and $K\,Sun + K\,moon = Sun + moon = moon$ ✓

If $K = 0$, then $K(Sun + moon) = K\,moon = 0\,moon = Sun$

also $K\,Sun + K\,moon = 0\,Sun + 0\,moon = Sun + Sun = Sun$.

Property 8 $k+\ell \neq 0$ and $k \neq 0$, $\ell \neq 0$.

$(k+\ell)$ moon $=$ moon

And k moon $+ \ell$ moon $=$ moon $+$ moon $=$ sun

Therefore, property 8 is Not satisfied.

Defining a Vector space of three elements.

Thm 5.2.1 Assume

1) $\vec{W} \subset \vec{V}$ with one or more vectors

2) V is a Vector space.

W is a Subspace of V if and only if

a) W closed under addition. For any $\vec{u}, \vec{v}$ in W

$\vec{u} + \vec{v}$ is also in W.

b) W is closed under scalar multiplication

For any $\vec{u}$ in V and α real

$\alpha \vec{u}$ is in V.

Proof.- ($\rightarrow$) Trivial (a) and (b) particular cases of all ten axioms.

($\leftarrow$) All properties (axioms) are inherited, except (4) and (5)

(4) There is a zero, since if $\vec{u}$ is in W

Thm 5.1.1 $\Rightarrow$ $0.\vec{u}$ is in W, but $0\vec{u} = \vec{0}$ ✓

(5) For any $\vec{u}$ in W, there is $-\vec{u}$, since

thm 5.1.1 $\Rightarrow$ $(-1)\vec{u}$ is in W, but $(-1)\vec{u} = -\vec{u}$ ✓

Definition:-

$\vec{W}$ is a linear Combination of the vectors $\vec{V}_1, .., \vec{V}_r$ if $\vec{W} = K_1 \vec{V}_1 + \cdots + K_r \vec{V}_r$

where $K_1, .. K_r$ are scalars.

Thm 5.2.3 If $\vec{V}_1, .., \vec{V}_r$ in V

a) W: set of all possible linear comb. of $\{\vec{V}_1 .., \vec{V}_r\}$ is a Subspace of V.

b) W is the smallest subspace of V containing $\vec{V}_1, .., \vec{V}_r$

(If $U \subset V$ and $\{\vec{V}_1 .., \vec{V}_r\} \subset U \Rightarrow W \subset U$.

Proof:- Easy.

Definition:- $S = \{\vec{V}_1, .., \vec{V}_r\}$ and W as in (a) thm 5.2.3

W is Called the space Spanned by S.

Notation: $W = \text{Span}(S)$ or $W = \text{Span}\{\vec{V}_1 .. \vec{V}_r\}$.

Thm 5.2.4 $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\}$, $S' = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$

two set of vectors in V (vector space),

then

$$\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\} = \text{span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$$

if and only if

Every vector in S is a linear comb. of vectors in S'

and

Every vector in S' is a linear Comb. of Vectors in S.

Example ~~7~~ 16.– (book) Subspaces of $\mathbb{R}^3$.

$$A\vec{x} = \vec{0} \quad \text{or} \quad \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the Soln. space.

Row reducing augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x - 2y + 3z = 0$$

solution space is this plane through orig.

$$A\bar{x} = 0, \text{ where } \begin{bmatrix} 1 & 5 & 7 & 0 \\ 2 & 4 & 2 & 0 \\ 3 & 2 & -5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 7 & 0 \\ 2 & 4 & 2 & 0 \\ 3 & 2 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} x = 3z \\ y = -2z \\ z \text{ free} \end{array}$$

Soln. space is a line through origin

$$x = 3t, \quad y = -2t, \quad z = t.$$

Example ~~12~~ 15. - (Sect ~~5.2~~ 4.2)

$\vec{V}_1 = (1,1,2)$, $\vec{V}_2 = (1,0,1)$, $\vec{V}_3 = (2,1,3)$

do not span $\mathbb{R}^3$.

If they do then any vector $\vec{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$

will be a lin. comb. of $\vec{V}_1, \vec{V}_2, \vec{V}_3$.

or

$$K_1 \vec{V}_1 + K_2 \vec{V}_2 + K_3 \vec{V}_3 = (b_1, b_2, b_3)$$

or

$$K_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + K_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + K_3 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

or

$$\begin{aligned} K_1 + K_2 + 2K_3 &= b_1 \\ K_1 \quad\quad + K_3 &= b_2 \\ 2K_1 + K_2 + 3K_3 &= b_3 \end{aligned}$$

or

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or

$$\boxed{A \vec{K} = \vec{b}} \quad (*)$$

Now,

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ det $A = 0$ $\Rightarrow$ Equiv. thm.

Nonhomog. syst. (*) is not consist. for some $\vec{b}$.

<u>Example ~~9.~~ 14</u> (Sect. ~~5.2~~ 4.2)

$\vec{u} = (1, 2, -1)$, $\vec{v} = (6, 4, 2)$ in $\mathbb{R}^3$

and $\vec{w} = (9, 2, 7)$, $\vec{w}' = (4, 2, 7)$

Prove that $\vec{w}$ is lin. comb. but w' is not

$$(9, 2, 7) = k_1(1, +2, -1) + k_2(6, 4, 2)$$

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned} \qquad \begin{bmatrix} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$k_1 = -3$, $k_2 = 2$.

For $\vec{w}'$, we have

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned} \quad \left| \quad \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 4 \\ 0 & -8 & 6 \\ 0 & 0 & 5 \end{bmatrix} \right.$$

inconsistent!

Does it mean $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is lin. indep?

Example ~~4~~ 2 – (Sect ~~5.3~~ 4.3)

$$\vec{V}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \vec{V}_2 = \begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix}, \quad \vec{V}_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Clearly, $\vec{V}_1 + \vec{V}_2 = \begin{pmatrix} 6 \\ 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 2\vec{V}_3$

The set $\{\vec{V}_1, \vec{V}_2, \vec{V}_3\}$ is lin. dep.

Procedure to determine lin. dep. condition.

$$k_1\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + k_2\begin{pmatrix} 5 \\ 6 \\ -1 \end{pmatrix} + k_3\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ equiv. to

$$\begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Bigg\| \quad A \sim \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow k_3 \text{ fr}$$

$A\vec{x} = \vec{0}$

$\Rightarrow k_1 = -\frac{1}{2}k_3,\ k_2 = -\frac{1}{2}k_3$