Outline for a final class of Circular membrane. (wave equ.)

1) Set up. IBVP.

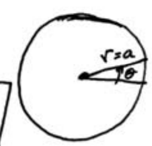
2) Sep. of Vabs: $u(r,\theta,t) = h(t)\,\phi(r,\theta)$

t-equ. : $h''(t) + \lambda c^2 h(t) = 0$

Solns: $\boxed{h(t) = A\cos c\sqrt{\lambda}t + B\sin c\sqrt{\lambda}t}$ for $\lambda \geq 0$

Space equ: Eigenvalue Problem:

$$\begin{cases} \nabla^2_{r,\theta}\phi + \lambda\phi = 0 \iff \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2} + \lambda\phi = 0 \\ \phi(a,\theta) = 0 \end{cases}$$



3) Further separation of Space Variables: $\boxed{\phi(r,\theta) = f(r)\,g(\theta)}$

Two Eigenvalue problems:

$$\begin{cases} \dfrac{d^2g}{d\theta^2} + \mu g(\theta) = 0 \\ g(-\pi) = g(\pi) \\ \dfrac{dg}{d\theta}(-\pi) = \dfrac{dg}{d\theta}(\pi) \end{cases}$$

regular S-L EVP.

$$\begin{cases} r\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) + (\lambda r^2 - f) = 0 \\ f(a) = 0 \\ |f(0)| < \infty \end{cases}$$

Not regular SL-EVP, but still Good properties. of regular problems.

4) Eigenvalues and Eigenfunctions for $g(\theta)$:

$$\mu_m = m^2, \quad m = 0,1,2,\ldots$$

$$g_m(\theta) = \begin{Bmatrix} \sin(m\theta) \\ \cos(m\theta) \end{Bmatrix}_{m=0}^{\infty}$$

5) Eigenvalues and Eigenfunctions for $f(r)$:

a) Transformation to Bessel's differential equation:

$$z = \sqrt{\lambda}\, r$$

$$r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} + (\lambda r^2 - m^2) f = 0$$

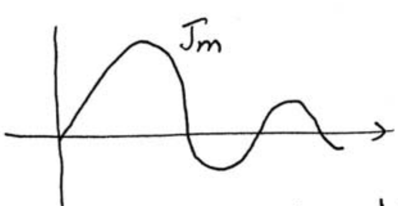
$$\downarrow$$

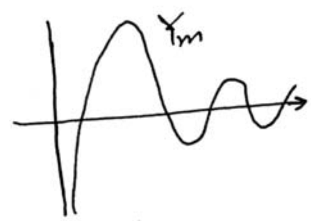
$$z^2 \frac{d^2 \hat{f}}{dz^2} + z \frac{d\hat{f}}{dz} + (z^2 - m^2)\hat{f} = 0$$

General solution: $\hat{f}(z) = C_1 J_m(z) + C_2 Y_m(z)$.

$$z = \sqrt{\lambda}\, r \longrightarrow$$

$$f(r) = C_1 J_m(\sqrt{\lambda}\, r) + C_2 Y_m(\sqrt{\lambda}\, r).$$





Since $|f(0)| < \infty$

$$\Rightarrow \boxed{f(r) = C_1 J_m(\sqrt{\lambda}\, r)}$$

B.c: $\quad \mathcal{U}(a,\theta,t)=0 \Rightarrow f(a)=0.$

$\therefore$ For each m,

$$0 = f(a) = C_1 J_m(\sqrt{\lambda}\, a) = 0$$

$$\Rightarrow \sqrt{\lambda}\, a = \text{"root of } J_m(z)\text{"}$$

$$\Rightarrow \underline{\sqrt{\lambda}\, a} = z_{mn}, \quad \begin{matrix} m \text{ fixed} \\ n=1,2,\ldots \end{matrix} \Rightarrow \boxed{\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2}$$

and Corresponding eigenfunctions are

$$\boxed{J_m(\sqrt{\lambda_{mn}}\, r) = J_m\left(\frac{z_{mn}}{a} r\right)}. \quad \begin{matrix} m \text{ fixed} \\ n=1,2,\ldots \end{matrix}$$

<u>DISCUSS COMPLETENESS AND ORTHOGONALITY</u> (page 8 of notes).

6) <u>Eigenvalues and Eigenfns. for 2-D EVP:</u>

$$\boxed{\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2}, \quad \boxed{\phi_{mn} = J_m\left(\frac{z_{mn}}{a} r\right) \begin{Bmatrix} \sin(m\theta) \\ \cos(m\theta) \end{Bmatrix} \begin{matrix} m=0,1,2\ldots \\ n=1,2,\ldots \end{matrix}}$$

7) <u>Product Solns:</u>

$$J_m(\sqrt{\lambda_{mn}}\, r) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} \begin{Bmatrix} \cos(c\sqrt{\lambda_{mn}}\, t) \\ \sin(c\sqrt{\lambda_{mn}}\, t) \end{Bmatrix}$$

8) <u>Final Series Solution:</u>

$$u(r,\theta,t) =$$

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) + B_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \sin(m\theta) \right] \cos\left(c\sqrt{\lambda_{mn}}\, t\right) +$$

$$+ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left[C_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \cos(m\theta) + D_{mn} J_m\left(\sqrt{\lambda_{mn}}\, r\right) \sin(m\theta) \right] \sin\left(c\sqrt{\lambda_{mn}}\, t\right)$$

7.7.9 Circularly Symmetric Case.

$$u(r,\theta,t) = u(r,t)$$

$$\frac{\partial}{\partial\theta} = 0. \Rightarrow \nabla^2_{r,\theta} u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}}$$

Then IBVP reduces to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{c^2}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) & (1.1) \\ & (1.2) \\ u(a,t) = 0 & \\ u(r,0) = \alpha(r), \quad \dfrac{\partial u}{\partial t}(r,0) = \beta(r) & (1.3) \end{cases}$$

Separating variables:

$$\boxed{u(r,t) = \phi(r)\,h(t)} \qquad (1.4)$$

$$\frac{d^2 h(t)}{dt^2}\phi(r) = \frac{c^2}{r} h(t)\frac{d}{dr}\left(r\frac{d\phi}{dr}\right)$$

dividing by $c^2 h(t)\phi(r)$.

$$\frac{1}{h(t)}\frac{1}{c^2}\frac{d^2 h}{dt^2} = \frac{1}{r\phi(r)}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda$$

Time equation:

$$\boxed{\frac{d^2 h}{dt^2} + \lambda c^2 h = 0} \qquad (1.5)$$

Eigenvalue Problem:

$$\begin{cases} \frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0 & (2.1) \\ \phi(a) = 0, \quad |\phi(0)| < \infty & (2.2) \end{cases}$$

A Rayleigh Quotient can be obtained for EVP (2.1)-(2.2) by multiplicating by $\phi(r)$ and $\int_0^a ( )\, r\,dr$. By doing this, we obtain

$$\lambda = \frac{\int_0^a r\left(\frac{d\phi}{dr}\right)^2 dr}{\int_0^a \phi^2(r)\, r\,dr} \geq 0$$

Prove this!

and from (2.2), $\lambda > 0$ only possibility.

Following a similar procedure as the one employed for the more general case, we introduce the new variable

$$\boxed{z = \sqrt{\lambda}\, r}$$

and equation (2.1) is transformed into Bessel's differential equation of order zero.

$$r^2 \frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \lambda r^2\phi = 0 \overset{z=\sqrt{\lambda}r}{\underset{\hat{\phi}(z)=\hat{\phi}(\sqrt{\lambda}r)=\phi(r)}{\longleftrightarrow}} z^2\frac{d^2\hat{\phi}}{dz^2} + z\frac{d\hat{\phi}}{dz} + z^2\hat{\phi} = 0.$$

The general solution of Bessel's ODE of order zero:

$$z^2 \frac{d^2\hat{\phi}}{dr^2} + z^2 \frac{d\hat{\phi}}{dr} + z^2 \hat{\phi} = 0$$

is given by

$$\boxed{\hat{\phi}(z) = C_1 J_0(z) + C_2 Y_0(z)}$$

And returning the change of variables:

$$\boxed{\phi(r) = C_1 J_0(\sqrt{\lambda} r) + C_2 Y_0(\sqrt{\lambda} r)}$$

Since $|\phi(0)| < \infty \Rightarrow C_2 = 0.$

and $\boxed{\phi(r) = C_1 J_0(\sqrt{\lambda} r)}$

Using the B.C. at $r = a$.

$$J_0(\sqrt{\lambda} a) = 0 \Rightarrow \sqrt{\lambda} a = z_{0n}, \quad n = 1, 2, \ldots$$

zeros of Bessel fn. of order zero.

$$\Rightarrow \left\{ \lambda_n = \left(\frac{z_{0n}}{a}\right)^2, \quad n = 1, 2, \ldots \right\} \text{ eigenvalues.}$$

and the eigenfunctions are

$$\left\{ J_0\left(\frac{z_{0n}}{a} r\right) \right\}_{n=1}^{\infty}.$$

The time equation (1.5) has the general solution: $(\lambda > 0)$

$$\boxed{h(t) = A_n \sin\left(\sqrt{\lambda_n}\, ct\right) + B_n \cos\left(\sqrt{\lambda_n}\, ct\right).} \quad (4.1)$$

Then, Product Solns. are given by

$$\left[A_n \sin\left(\frac{z_{0n}}{a} ct\right) + B_n \cos\left(\frac{z_{0n}}{a} ct\right)\right] J_0\left(\frac{z_{0n}}{a} r\right)$$

Using the pple of Superposition the final Series soln. is given by

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{z_{0n}}{a} ct\right) + B_n \cos\left(\frac{z_{0n}}{a} ct\right)\right] J_0\left(\frac{z_{0n}}{a} r\right)}$$

A_n and B_n are determined using the orthogonality (prove it!) of the Bessel's function of order zero.

In fact, $$\alpha(r) = u(r,0) = \sum_{n=1}^{\infty} B_n J_0\left(\frac{z_{0n}}{a} r\right)$$

multiplying by $J_0\left(\frac{z_{0m}}{a} r\right) r$ and $\int_0^a dr$ and Solving for B_n.

$$B_n = \frac{\int_0^a \alpha(r) J_0\left(\frac{z_{0n}}{a} r\right) r\, dr}{\int_0^a J_0^2\left(\frac{z_{0n}}{a} r\right) r\, dr}, \quad n = 1, 2, \ldots$$

Similarly, A_n's are obtained.