3.6 Complex form of F.S.

Show That

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi}{L}x} \qquad (1)$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{in\pi}{L}x}\, dx. \qquad (2)$$

From

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(\,) + b_n \sin(\,) \qquad (3)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f\, dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f \cos(\,)\, dx, \qquad (4)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f \sin(\,)\, dx.$$

Proof. Use Euler's form.

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \qquad (5)$$

Substituting (5) into (3)

$$f(x) \sim a_0 + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n\left(e^{\frac{in\pi}{L}x}+e^{-\frac{in\pi}{L}x}\right) + b_n\left(\frac{e^{\frac{in\pi}{L}x}-e^{-\frac{in\pi}{L}x}}{i}\right)\right]$$

$$\Rightarrow \quad f(x) \sim a_0 + \frac{1}{2} \sum_{n=1}^{\infty} \left[(a_n - i b_n)\, e^{i \frac{n\pi}{L} x} + (a_n + i b_n)\, e^{-i \frac{n\pi}{L} x} \right]$$

$$\sim a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - i b_n)\, e^{i \frac{n\pi}{L} x} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + i b_n)\, e^{-i \frac{n\pi}{L} x}$$

If $\quad n \longrightarrow -n$

$$f(x) \sim a_0 + \frac{1}{2} \sum_{n=-1}^{-\infty} (a_{-n} - i b_{-n})\, e^{-i \frac{n\pi}{L} x} + \frac{1}{2} \bigcirc\, e^{-i \frac{n\pi}{L} x}$$

but $\quad a_{-n} = a_n \;,$ from (4.1)

$$\boxed{b_{-n} = -b_n \;, \text{ from } (4.3).}$$

then defining $\quad c_0 = a_0 \;, \quad c_n = \dfrac{a_n + i b_n}{2}$

$$f(x) \sim c_0 + \sum_{n=-1}^{-\infty} c_n\, e^{-i \frac{n\pi}{L} x} + \sum_{n=1}^{\infty} c_n\, e^{-i \frac{n\pi}{L} x}$$

$$\sim c_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} c_n\, e^{-i \frac{n\pi}{L} x} = \sum_{n=-\infty}^{\infty} c_n\, e^{-i \frac{n\pi}{L} x}$$

Therefore,

$$\boxed{f(x) \sim \sum_{n=-\infty}^{\infty}{}' c_n e^{-i\frac{n\pi}{L}x}}$$

where
$$c_n = \frac{1}{2}(a_n + i b_n) =$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\frac{n\pi}{L}x + i\sin\frac{n\pi}{L}x\right)dx$$

$\therefore$

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\frac{n\pi}{L}x}\,dx}$$