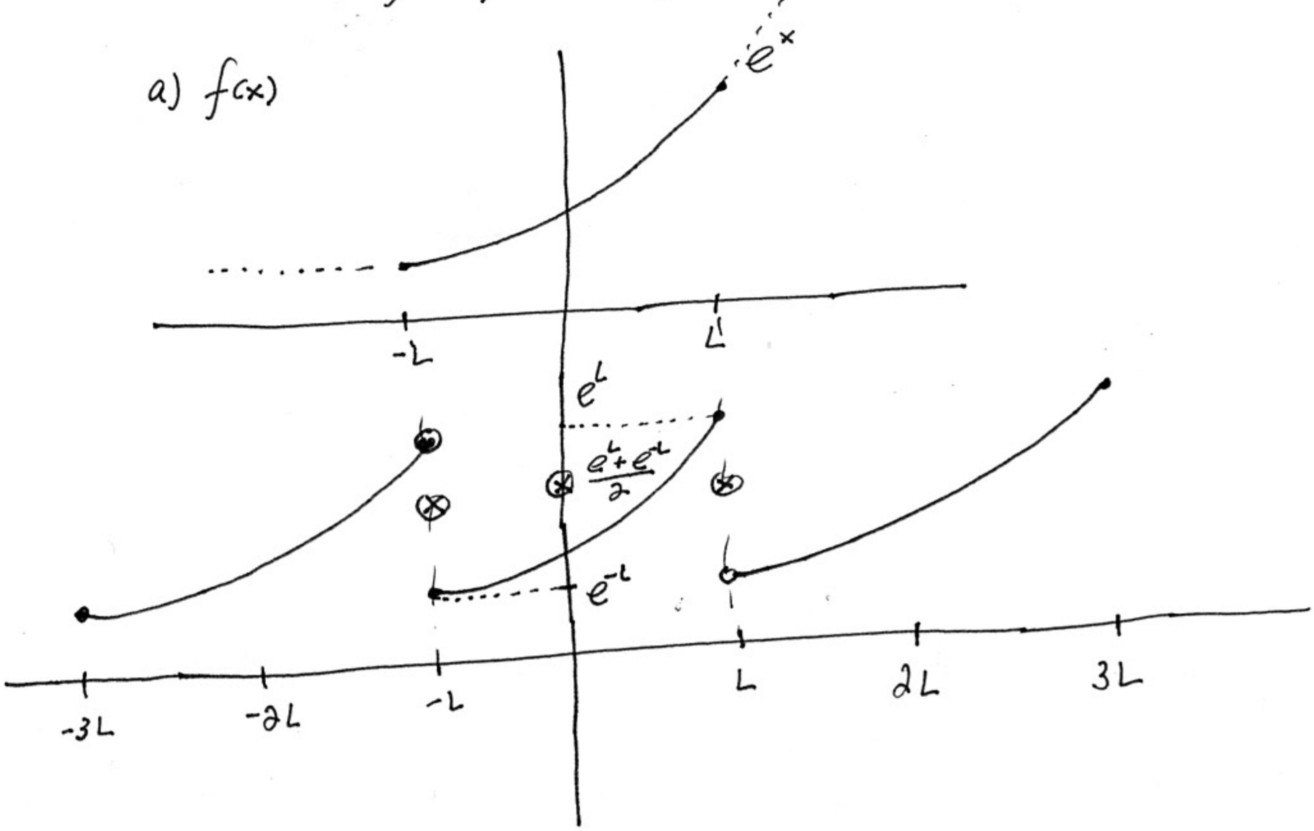
Review of Fourier Series using $f(x) = e^x$.

I) Fourier Series of $f(x) = e^x, \quad -L \leq x \leq L$.

a) $f(x)$



$$e^x \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \quad x \in (-\infty, \infty)$$

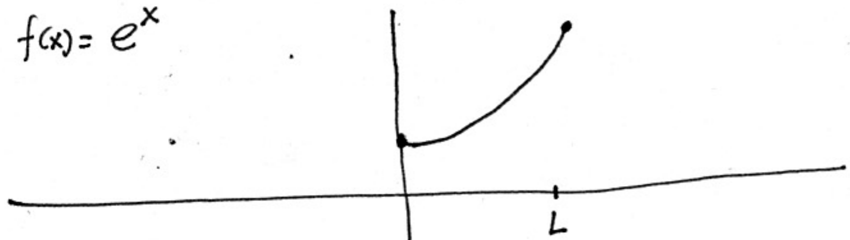
$$a_0 = \frac{1}{2L}\int_{-L}^{L} e^x dx = \frac{1}{2L}\left(e^L + e^{-L}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} e^x \cos\left(\frac{n\pi}{L}x\right) dx = \frac{(-1)^n L\left[-1+e^{2L}\right]}{L^2+(n\pi)^2}$$

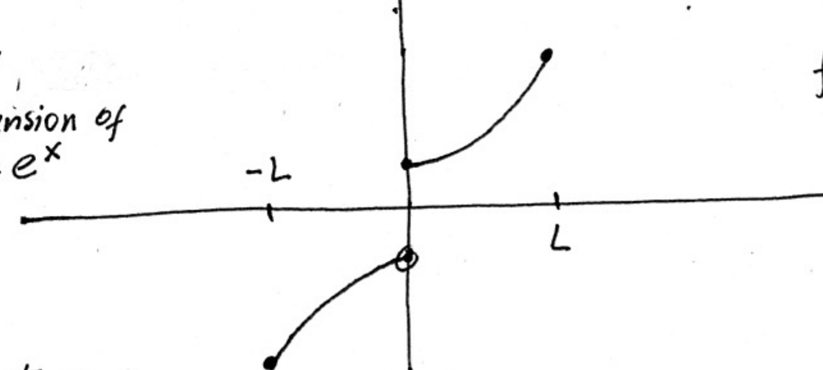
$$b_n = \frac{1}{L}\int_{-L}^{L} e^x \sin\left(\frac{n\pi}{L}x\right) dx = \frac{n\pi(-1)^n\left[1-e^{2L}\right]}{L^2+(n\pi)^2}$$

II) Fourier Series of Sin's of $f(x) = e^x, \quad 0 \le x \le L.$

a) $f(x) = e^x$

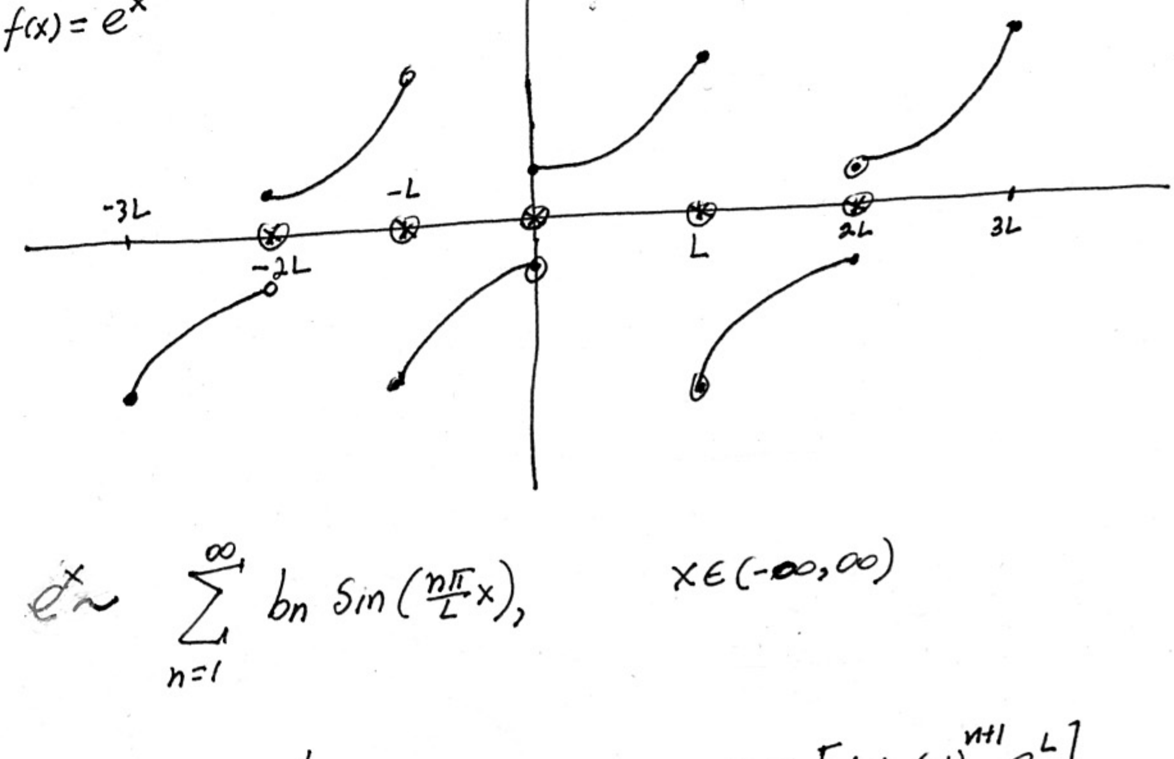


b) Odd extension of $f(x) = e^x$



$$f(x) = \begin{cases} e^x, & 0 < x < L \\ -e^{-x}, & -L \le x < 0 \end{cases}$$

c) Periodic extension of the odd extension of $f(x) = e^x$

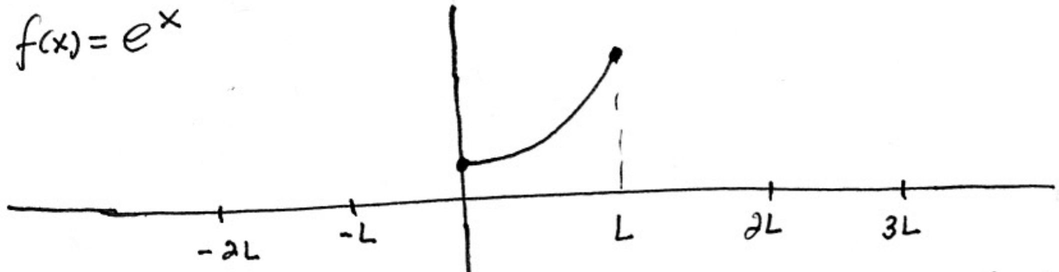


$$e^x \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \qquad x \in (-\infty, \infty)$$

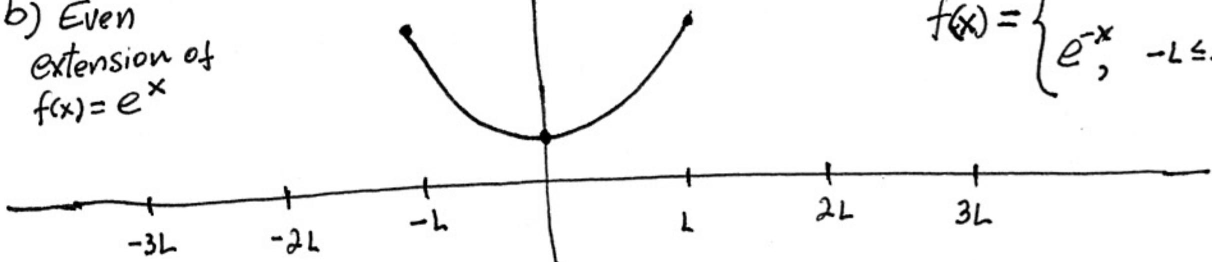
$$b_n = \frac{2}{L}\int_0^L e^x \sin\left(\frac{n\pi}{L}x\right) dx = \frac{n\pi\left[1 + (-1)^{n+1} e^L\right]}{L^2 + (n\pi)^2}$$

III) Fourier Series of Cosines of $f(x) = e^x$; $0 \le x \le L$.

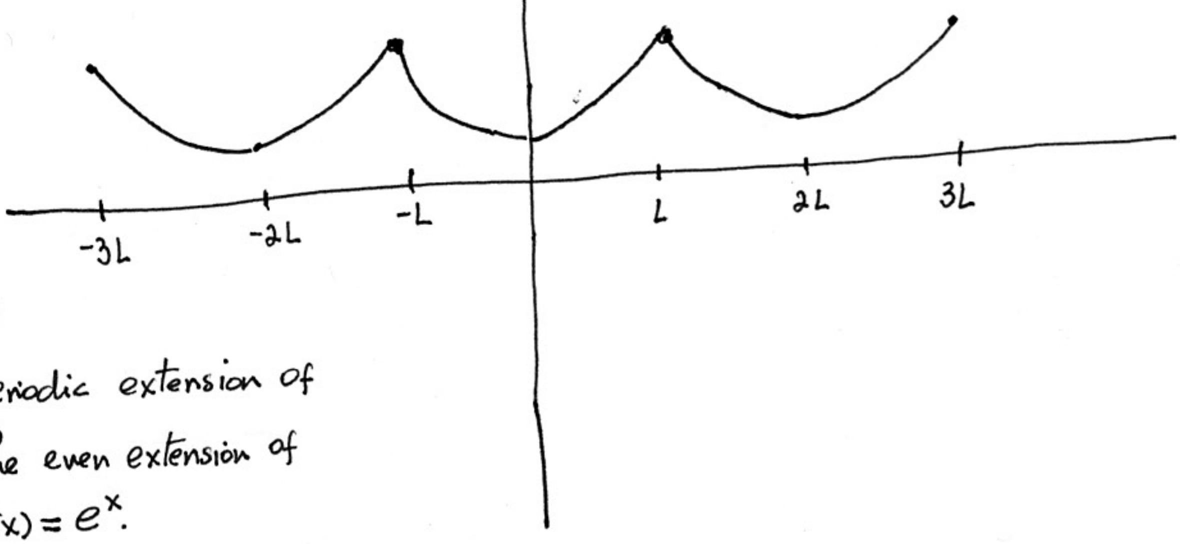
a) $f(x) = e^x$



b) Even extension of $f(x) = e^x$



$$f(x) = \begin{cases} e^x, & 0 \le x \le L \\ e^{-x}, & -L \le x < 0 \end{cases}$$



c) Periodic extension of the even extension of $f(x) = e^x$.

$$e^x \sim = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right)$$

$$a_0 = \frac{1}{L} \int_0^L e^x \, dx = \frac{e^L - 1}{L}$$

$$a_n = \frac{2}{L} \int_0^L e^x \cos\left(\frac{n\pi}{L} x\right) dx = \frac{L\left[(-1)^n e^L - 1\right]}{L^2 + (n\pi)^2}$$

3.3.4 Even and odd parts of $f(x)$

Given any function $f(x)$, it's always possible to express $f(x)$ as

$$f(x) = \frac{1}{2}\left[f(x) + f(-x) \right] + \frac{1}{2}\left[f(x) - f(-x) \right] \qquad (5.1)$$

Def.- The functions

$$f_o(x) \overset{def}{\equiv} \frac{1}{2}\left[f(x) + f(-x) \right] \qquad (5.2)$$

and $$f_e(x) \equiv \frac{1}{2}\left[f(x) - f(-x) \right] \qquad (5.3)$$

are called the even part of $f(x)$ and the odd part of $f(x)$, respectively.

Thm.- If $f(x)$, $-L \le x \le L$ is piecewise smooth, then

$$f_e(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

and $$f_o(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

where a_n, b_n are the coefficient of the Fourier Series of $f(x)$ over the interval $[-L, L]$.

Proof – Fourier series of $f(x)$ is given by

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow f(-x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) - \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

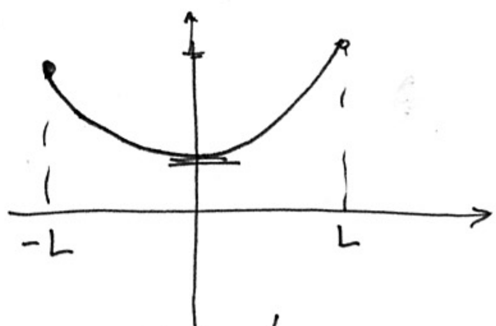
$$\Rightarrow f_e(x) = \frac{1}{2}\left[f(x) + f(-x)\right] \sim \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)}_{\text{Even part}} \qquad (6.1)$$

and

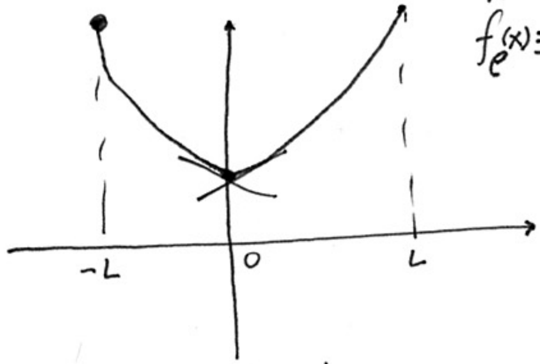
$$\Rightarrow f_o(x) = \frac{1}{2}\left[f(x) - f(-x)\right] \sim \underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)}_{\text{Odd part}} \qquad (6.2)$$

Example: For $f(x) = e^x$; $\quad -L \le x \le L$

$$f_e(x) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$



part

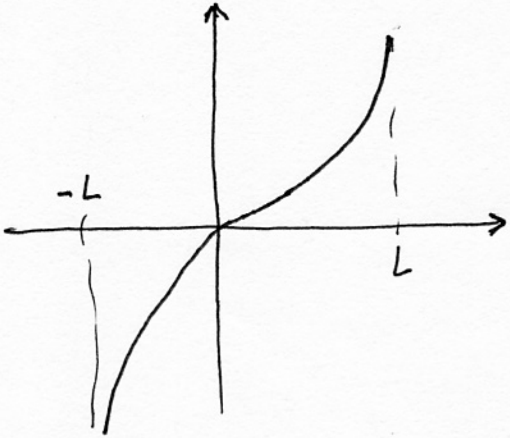


$$\hat{f}_e(x) \equiv \begin{cases} e^x, & 0 \le x \le L \\ e^{-x}, & -L \le x \le 0 \end{cases}$$

extension

odd part

$$f_o(x) = \frac{e^x - e^{-x}}{2} = \sinh(x)$$



Odd extension

$$\hat{f}_o(x) \equiv \begin{cases} e^x, & 0 \le x \le L \\ -e^{-x}, & -L < x < 0 \end{cases}$$