7.5 Green's Formula. Self-adjoint operators and multidimensional EVP's.

For the 1-D Sturm–Liouville operator

$$L(u) = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + qu$$

We obtained two important formulas:

a) Lagrange's identity:

$$\boxed{uLv - vLu = \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]}$$

b) Green's formula:

$$\boxed{\int_a^b [uLv - vLu]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b}$$

For the 2-D or 3-D multidimensional EVP defined by the Laplace operator, $L \equiv \nabla^2$. There are similar formulas:

Multidimensional Version of Lagrange's identity:

a) $$\boxed{u\nabla^2 v - v\nabla^2 u = \nabla\cdot(u\nabla v - v\nabla u)} \qquad (2.1)$$

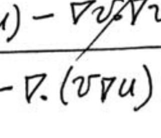
b) Green's formula:

2-D: $$\iint_{\Omega}\left[u\nabla^2 v - v\nabla^2 u\right]dv = \oint_{\partial\Omega}\left[u\nabla v - v\nabla u\right]\cdot\hat{n}\,ds \qquad (2.2)$$

Ω is a volume bounded by a sufficiently smooth surface $\partial\Omega$. Also, u and v are continuous together with their first derivatives inside $\Omega\cup\partial\Omega$ and they have conts. second derivatives in Ω.

(2.3)

3-D: $$\iiint_{\Omega}\left[u\nabla^2 v - v\nabla^2 u\right]dv = \iint_{\partial\Omega}\left[u\nabla v - v\nabla u\right]\cdot\hat{n}\,ds.$$



Derivation of (2.1):

$$\nabla\cdot(u\nabla v) = u\nabla^2 v + \nabla u\cdot\nabla v \Rightarrow u\nabla^2 v = \nabla\cdot(u\nabla v) - \cancel{\nabla u\cdot\nabla v}$$

$$\nabla\cdot(v\nabla u) = v\nabla^2 u + \nabla v\cdot\nabla u \Rightarrow v\nabla^2 u = \nabla\cdot(v\nabla u) - \cancel{\nabla v\cdot\nabla v}$$

$$u\nabla^2 v - v\nabla^2 u = \nabla\cdot(u\nabla v) - \nabla\cdot(v\nabla u)$$

or $$\boxed{u\nabla^2 v - v\nabla^2 u = \nabla\cdot(u\nabla v - v\nabla u)}$$

Divergence Theorem.

Concept of Region:- An open set containing all or some of the points forming its boundary.

Def.- A closed surface $\partial\Omega$ that consists of a finite number of smooth pieces joined together at the boundaries (curves defining boundaries) is called piecewise-smooth surface.

Def.- By a smooth surface S, we mean

$$S: \ \vec{r} = \vec{r}(u,v), \quad (u,v) \in D.$$

Such that $\vec{r}(u,v)$ is continuously differentiable and its unit normal vector $\hat{n}(\vec{x}_s)$ is continuous on S.

Theorem.- a) Ω is a bounded region.

b) $\partial\Omega$ is the closed piecewise smooth surface of Ω.

c) $F(\vec{x})$ is continuous on $\Omega \cup \partial\Omega$.

d) $F(\vec{x})$ is continuosly differentiable in Ω.

e) $\hat{n}(\vec{x}_s)$ is the unit outer normal vector to Ω at $\vec{x}_s$.

Then,

$$\iiint_{\Omega} \nabla \cdot \vec{F}(\vec{x})\, dv = \iint_{\partial\Omega} (\vec{F}\cdot\hat{n})(\vec{x}_s)\, ds.$$

Derivation of Green's formula:

3-D: Integrating (2.1)

$$\iiint_{\Omega}\left[u\nabla^2 v - v\nabla^2 u\right]dv = \iiint_{\Omega}\nabla\cdot(u\nabla v - v\nabla u)\,dv$$

$$\overset{\text{Gauss Thm}}{=\!=} \iint_{\partial\Omega}(u\nabla v - v\nabla u)\cdot\hat{n}\,ds \quad ✓$$

2-D:

$$\iint_{\Omega}\left[u\nabla^2 v - v\nabla^2 u\right]dv = \iint_{\Omega}\nabla\cdot(u\nabla v - v\nabla u)\,dv =$$

$$\overset{\text{Gauss Thm}}{=\!=} \oint(u\nabla v - v\nabla u)\cdot\hat{n}\,ds \quad ✓$$

Definition.- If u and v are ① continuous together with their first derivatives inside $\Omega\cup\partial\Omega$, ② they have conts second derivatives in Ω, and ③ they satisfy the same set of boundary conditions of the regular Sturm-Liouville type, then

$$\iint_{\partial\Omega}(u\nabla v - v\nabla u)\cdot\hat{n}\,ds = 0$$

$\Rightarrow \iiint_{\Omega}\left[u\nabla^2 v - v\nabla^2 u\right]dv = 0$. Then, it's said that the Laplacian is a self-adjoint multidimensional operator.
$L \equiv \nabla^2$

We have said before, that if u and v satisfy

$$\beta_1 u + \beta_2 \nabla u.\hat{n}\,(\vec{x}_s) = 0, \qquad \vec{x}_s \in \partial\Omega.$$

$$\Rightarrow \iint_{\partial\Omega} (u\nabla v - v\nabla u).\hat{n}\, ds = 0.$$

Proof:- If ① $\beta_2 = 0$ Dirichlet BC's.
$\beta_1 \neq 0 \Rightarrow u(\vec{x}_s) = 0, \quad v(\vec{x}_s) = 0$

$$\Rightarrow \iint_{\partial\Omega} (\cancel{u}^{\,0}\nabla v - \cancel{v}^{\,0}\nabla u).\hat{n}\, ds = 0$$

If ② $\beta_2 \neq 0$, $\beta_1 = 0$ $\Rightarrow$ Neumann Cond. $\nabla u.\hat{n}\,(\vec{x}_s) = 0$, $\nabla v.\hat{n}(\vec{x}_s) = 0$

$$\Rightarrow \iint_{\partial\Omega} [u\, \cancel{\nabla v.\hat{n}}^{\,0} - v\, \cancel{\nabla u.\hat{n}}^{\,0}]\, ds = 0$$

If ③ $\beta_1 \neq 0$, $\beta_2 \neq 0$. $\Rightarrow \begin{cases} u = \dfrac{-\beta_2}{\beta_1} \nabla u.\hat{n} \\ v = -\dfrac{\beta_2}{\beta_1} \nabla v.\hat{n}. \end{cases}$

$$\Rightarrow \iint_{\Omega} [u\nabla v.\hat{n} - v\nabla u.\hat{n}]\, ds =$$

$$= \iint_{\Omega} \left[\left(-\frac{\beta_2}{\beta_1} \nabla u.\hat{n}\right) \nabla v.\hat{n} - \left(-\frac{\beta_2}{\beta_1} \nabla v.\hat{n}\right) \nabla u.\hat{n}\right] ds \overset{\text{clearly}}{=} 0$$

work also in BC's. where $\beta_2 \neq 0$ on part of $\partial\Omega$ and $\beta_2 = 0$ on another part.

7.6 Rayleigh Quotient and Laplace's Equation.

Consider the Sturm-Liouville multidimensional equation:

$$\nabla^2\phi + \lambda\phi = 0, \qquad \vec{x} \in \Omega$$

multiplying by ϕ and integrating over Ω.

$$\iiint_\Omega \phi\nabla^2\phi \, dv + \iiint_\Omega \lambda\phi^2 \, dv = 0$$

$$\Rightarrow \quad \boxed{\lambda = \frac{-\iiint_\Omega \phi\nabla^2\phi \, dv}{\iiint_\Omega \phi^2 \, dv}} \qquad (5.1)$$

Integration by parts for multidimensional functions:

(1-D) $\quad \dfrac{d}{dx}(fg) = f\dfrac{dg}{dx} + \dfrac{df}{dx}g$

$\begin{pmatrix}\text{2-D} \\ \text{or} \\ \text{3-D}\end{pmatrix}$ If $f : \mathcal{D} \subset \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \to f(x,y,z)$ $\qquad \vec{G} : \mathcal{D} \subset \mathbb{R}^3 \to \mathbb{R}^3$, $(x,y,z) \to (g_1(x,y,z), g_2(x,y,z), g_3(x,y,z))$

$$\nabla\cdot(f\vec{G}) = f\,\nabla\cdot\vec{G} + \nabla f\cdot\vec{G}$$

If $f = \phi$ and $\vec{G} = \nabla\phi$, then

$$\boxed{\nabla\cdot(\phi\nabla\phi) = \phi\,\nabla\cdot(\nabla\phi) + \nabla\phi\cdot\nabla\phi = \phi\nabla^2\phi + \nabla\phi\cdot\nabla\phi.}$$

Or

$$\boxed{\nabla\cdot(\phi\nabla\phi) = \phi\,\nabla^2\phi + |\nabla\phi|^2} \qquad (6.1)$$

Solving for $\phi\nabla^2\phi$

$$\phi\,\nabla^2\phi = \nabla\cdot(\phi\nabla\phi) - |\nabla\phi|^2.$$

Therefore,
$$\iiint_\Omega \phi\nabla^2\phi\,dv \overset{I.P.}{=} \iiint_\Omega \nabla\cdot(\phi\nabla\phi)\,dv - \iiint_\Omega |\nabla\phi|^2\,dv$$

$$\xrightarrow{\text{Gauss Thm}} \iint_{\partial\Omega} \phi\nabla\phi\cdot\hat{n}\,ds - \iiint_\Omega |\nabla\phi|^2\,dv.$$

Substitution in (5.1).

$$\boxed{\lambda = \frac{-\iint_{\partial\Omega}\phi\nabla\phi\cdot\hat{n}\,ds + \iiint_\Omega |\nabla\phi|^2\,dv}{\iiint_\Omega \phi^2\,dv}} \qquad (6.1)$$

Application. The EVP

$$\begin{cases} \nabla^2\phi + \lambda\phi = 0, \quad \vec{x} \in \Omega & (7.1) \\ \text{with Dirichlet B.C.} \quad \phi(\vec{x}_s) = 0, \quad \vec{x}_s \in \partial\Omega. & (7.2) \end{cases}$$

has only positive eigenvalues.

Substituting the B.C. $\phi = 0$ in (6.1). results

$$\lambda = \frac{\iiint_\Omega |\nabla\phi|^2 dv}{\iiint_\Omega \phi^2(x) dx} \geq 0$$

If $\lambda = 0$, then $\nabla\phi = 0 \Rightarrow \frac{\partial\phi}{\partial x} = 0, \ \frac{\partial\phi}{\partial y} = 0$

$\Rightarrow \phi(\vec{x}) \equiv K$ (const), but $\phi(\vec{x}_s) = 0, \ \vec{x}_s \in \partial\Omega$

$\Rightarrow \phi(\vec{x}) \equiv 0$. Contradiction! $\Rightarrow \boxed{\lambda > 0}$

Consider the IBVP:

$$\begin{cases} u_t = K\nabla^2 u, \quad \vec{x} \in \Omega \\ u(\vec{x}_s, t) = 0, \quad \vec{x}_s \in \partial\Omega \\ u(\vec{x}, 0) = f(\vec{x}). \end{cases} \xrightarrow{\text{Equil.}} \begin{cases} \nabla^2 \hat{u} = 0 \\ \hat{u}(\vec{x}_s) = 0 \end{cases}$$

The Soln. for this equilibrium problem is $\boxed{u(x) \equiv 0,}$ $\vec{x} \in \Omega$.

If not

$$u(\vec{x}) \not\equiv 0$$

and $\lambda = 0$ would be an eigenvalue of (7.1), (7.2) with eigenfn. $\phi = u(x)$.

which we have already proved that is not possible.

If the IBVP has insulated BC's. : $\boxed{\nabla u . \hat{n} (\vec{x}_s) = 0,}$ $\vec{x}_s \in \partial\Omega$.

then from the Rayleigh quotient

$$u(x) \equiv k \text{ (const)}$$

Because, this is the only possible eigenfunction (lin. indep.) corresponding to $\lambda = 0$.