CHAPTER 12 (Haberman)

The Method of Characteristics.

First order Equations.

Consider the 1-D wave equation:

$$\boxed{u_{tt} - c^2 u_{xx} = 0} \qquad (1)$$

It can be written as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$$

Since

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2} + \cancel{c\frac{\partial^2 u}{\partial x \partial t}} - \cancel{c\frac{\partial^2 u}{\partial t \partial x}} - c^2\frac{\partial^2 u}{\partial x^2}$$

$$= u_{tt} - c^2 u_{xx} \checkmark = 0$$

Also, as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0.$$

If we call $\omega = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$

$$\Rightarrow \boxed{\frac{\partial \omega}{\partial t} + c\frac{\partial \omega}{\partial x} = u_{tt} - c^2 u_{xx}} \qquad (2)$$

The equation at the left is called a "first order linear wave equation" ($c = $ constant).
Now, we will learn how to solve an IVP for this type of equations.

First order wave equation

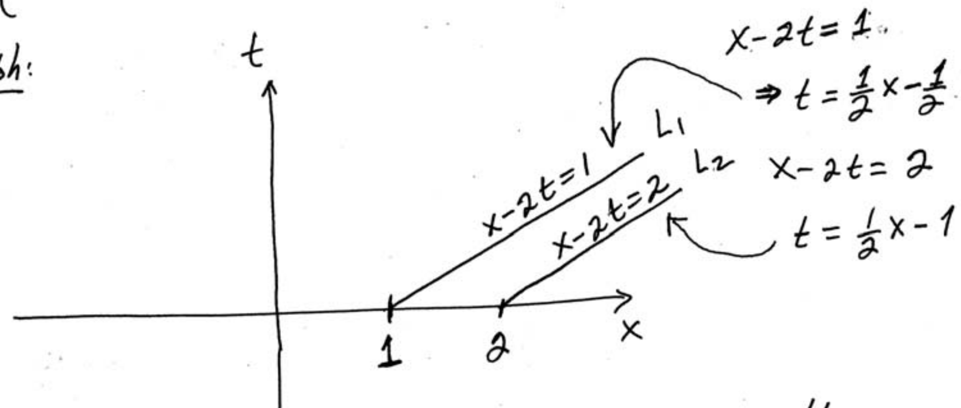
$$\begin{cases} u_t + a\,u_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

$$\boxed{u(x,t) = \phi(x-at)}$$ Any "a" and any I.C. $\phi(x)$.

Particular case:

$$\begin{cases} u_t + 2u_x = 0 \\ u(x,0) = 4x \end{cases} \Rightarrow u(x,t) = 4(x-2t).$$

Graph:



on L_1: $\quad u(x,t) = 4(x-2t) = 4*1 = 4.$

on L_2: $\quad u(x,t) = 4(x-2t) = 4*2 = 8$

Discuss wave behavior.

We can show now that

$$u(x,t) = \phi(x-at)$$

is indeed a Soln. of (1)-(2) if $\phi(x)$ has first derivatives. It might be points (x,t) where the Soln of (1) does not exist (jump discontinuity for example).

In fact,

$$u_t = \phi'(x-at)\,\frac{d}{dt}(x-at) = \phi'(x-at)(-a)$$

$$u_x = \phi'(x-at)\,\frac{d}{dx}(x-at) = \phi'(x-at)\,(1)$$

$$\Rightarrow \quad u_t + a\,u_x = -a\,\phi'(x-at) + a\,\phi'(x-at) = 0. \checkmark$$

Also I.C. is Satisfied since

$$u(x,0) = \phi(x-a0) = \phi(x). \quad \checkmark$$

<u>Application:</u> Solve the IVP.

$$\begin{cases} u_t + 2u_x = 0, & -\infty < x < \infty \\ u(x,0) = \begin{cases} 4x, & 0<x<1 \\ 0, & x<0 \text{ or } x>1 \end{cases} \end{cases}$$

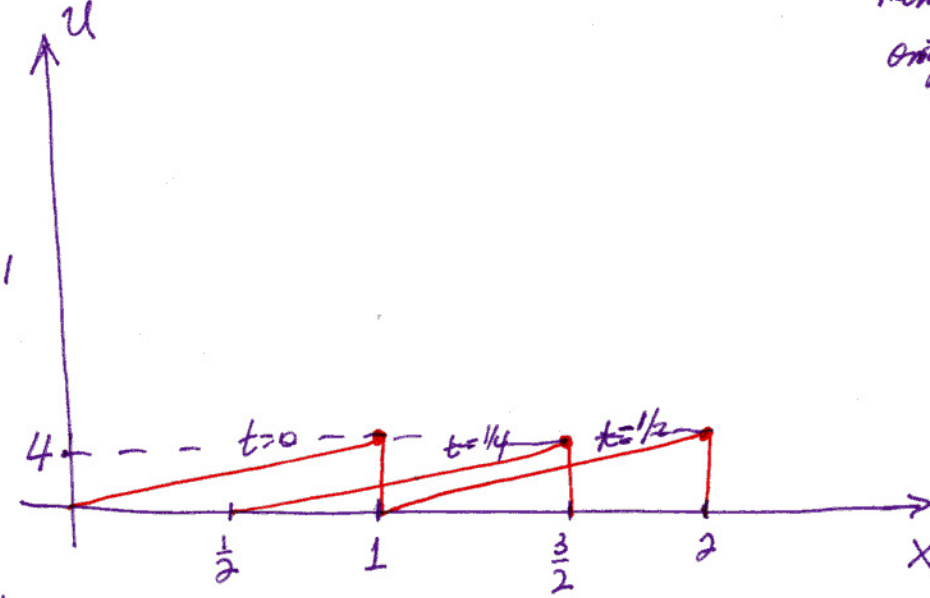
IVP

$$\begin{cases} u_t + 2u_x = 0 \\ u(x,0) = \begin{cases} 4x, & 0<x<1 \\ 0, & \text{otherwise.} \end{cases} \end{cases}$$

Explain using Floor of room. Select a corner as the origin of coords.

Soln:

$$u(x,t) = \begin{cases} 4(x-2t), & 0<x-2t<1 \\ 0, & \text{otherwise} \end{cases}$$



at $t=1/2$, $0<x-2(\frac{1}{2})<1$

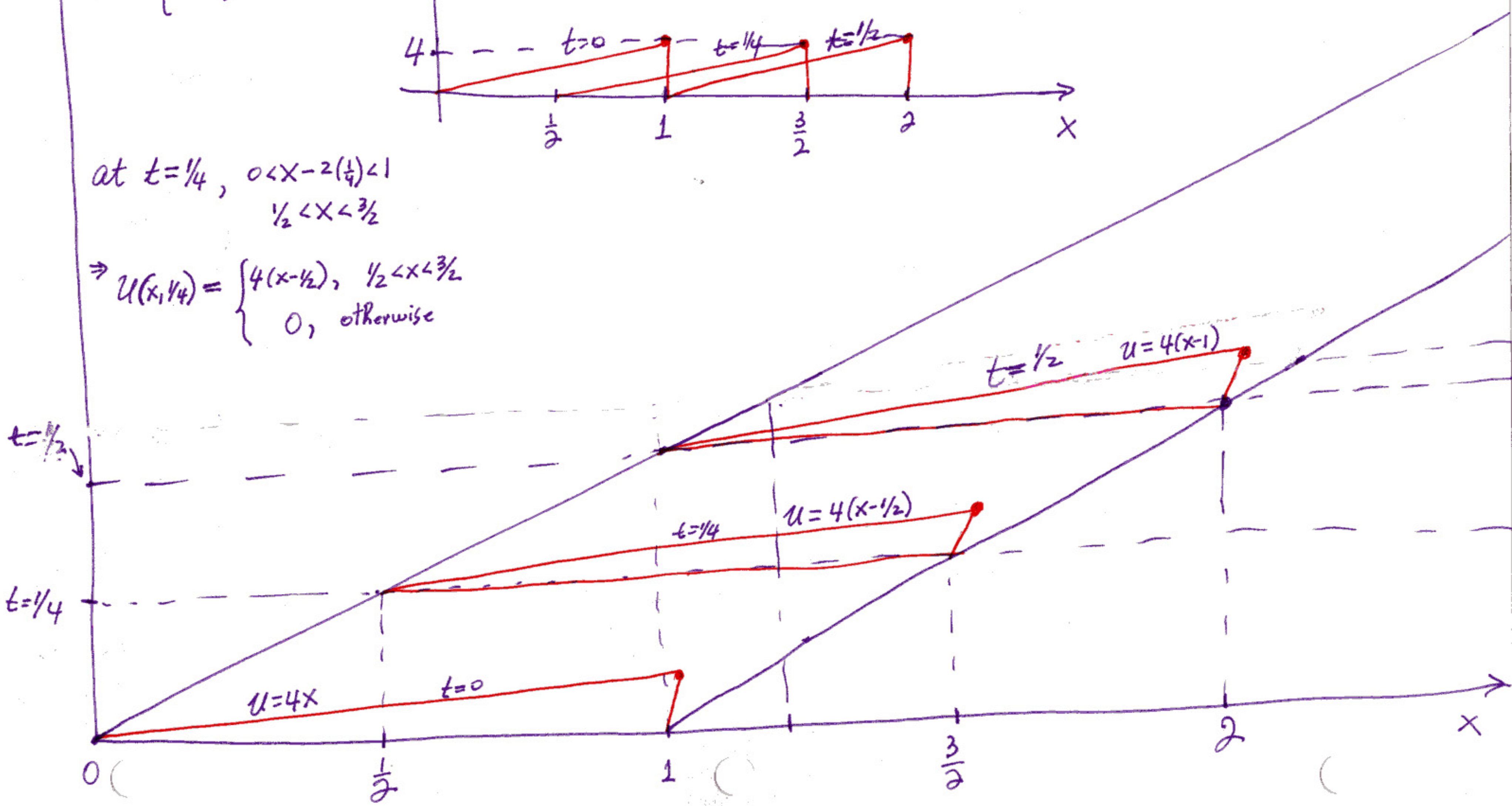
$1<x<2$

$$u(x,t) = \begin{cases} 4(x-1), & 1<x<2 \\ 0, & \text{otherwise} \end{cases}$$

at $t=1/4$, $0<x-2(\frac{1}{4})<1$

$1/2<x<3/2$

$$\Rightarrow u(x,1/4) = \begin{cases} 4(x-1/2), & 1/2<x<3/2 \\ 0, & \text{otherwise} \end{cases}$$



Method of Characteristic. First order PDE.

Consider the advection equation:

$$\begin{cases} u_t + a u_x = 0, & a>0, \quad -\infty<x<\infty, \quad t>0 \quad (1) \\ u(x,0) = \phi(x), & -\infty<x<\infty, \quad \phi \text{ is differentiable or even piecewise smooth} \quad (2) \end{cases}$$

Thm.- The soln. of Initial Value problem (1)-(2) is given by

$$u(x,t) = \phi(x-at), \quad -\infty<x<\infty, \quad t>0.$$

Proof.-

Assume ① $u(x,t)$ is a soln of (1)-(2).

② The point (x^*, t^*) is an arbitrary point in the semiplane $D=\{(x,t): -\infty<x<\infty, t\geq 0\}$.

③ Consider a curve $\mathcal{C}: x = x(t)$ passing through (x^*, t^*).

Then, along the curve $\mathcal{C}$,

$$\boxed{u(x,t) = u(x(t),t) = \hat{u}(t)} \quad (2.2)$$

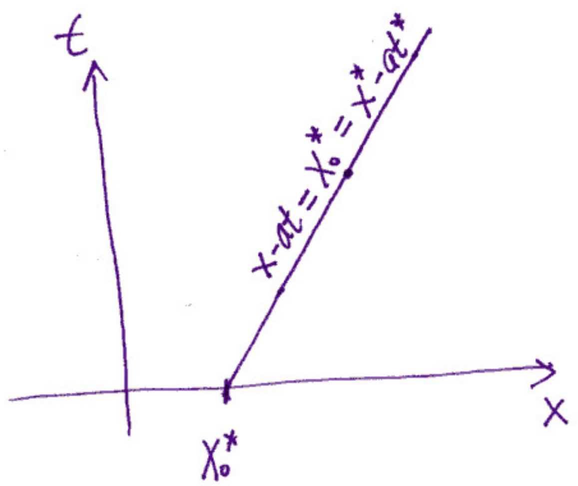
and

$$\frac{d\hat{u}}{dt}(t) \overset{\text{chain rule}}{=\!=} \frac{\partial u}{\partial x}(x(t),t)\,\frac{dx}{dt}(t) + \frac{\partial u}{\partial t}(x(t),t) \quad (3)$$

If we require $\mathcal{C}$ to satisfy $\boxed{\frac{dx}{dt}(t) = a} \overset{(3.1)}{\Rightarrow} \boxed{x(t) = at + x_0^*}$

In other words, $\mathcal{C}$ is the line: $\boxed{x - at = x_0^*}$ (3.2)

Since $(x^*, t^*) \in \mathcal{C} \Rightarrow \boxed{x^* - at^* = x_0^*}$ (3.3)



Combining (3) and (3.1), we are led to

$$\frac{d\hat{u}}{dt}(t) = \frac{\partial u}{\partial x}(x(t),t)\,\frac{dx}{dt}(t) + \frac{\partial u}{\partial t}(x(t),t) =$$

$$= a\,\frac{\partial u}{\partial x}(x(t),t) + \frac{\partial u}{\partial t}(x(t),t) \overset{\text{from (1)}}{=\!=} 0$$

Thus, $\boxed{\hat{u}(t) \equiv K}$, (4.1)

Using (2.2), we arrive to

$\boxed{u(x,t) = \hat{u}(t) = K,}$ for all $(x,t) \in \mathcal{C}$. (4.11)

It means the solution $u(x,t)$ is constant along $\mathcal{C}$.

Since $x - at = x_0^*$ on $\mathcal{C}$ $\Rightarrow$ at $t=0$, $x = x_0^*$

Then, $(x_0^*, 0) \in \mathcal{C}$ and $\boxed{u(x_0^*, 0) = K}$ (4.2)

But also, using the I.C. (2)

$\boxed{u(x_0^*, 0) = \phi(x_0^*)}$ (4.3)

From (4.3), (4.2), and (3.2)

$$K = \phi(x_0^*) = \phi(x - at), \quad \text{for all } (x,t) \in \mathcal{C}.$$

Subst. into (4.1) leads to

$$\boxed{u(x,t) = \phi(x - at),} \quad \text{for all } (x,t) \in \mathcal{C}$$

In particular,

$$\boxed{u(x^*, t^*) = \phi(x^* - at^*),} \quad \text{for our arbitrary } (x^*, t^*) \in D$$

Semiplane.

Therefore, $\boxed{u(x,t) = \phi(x - at),}$ for all $(x,t) \in D$.

Def.: The Line L: $x - at = x_0^*$ is Called a Characteristic line for the IVP (1)-(2).

Rmk: For any point (x,t) in the Semiplane D, there is a characteristic line passing through it.

12.2.5

$$\begin{cases} w_t + t\,w_x = 0 \\ w(x,0) = f(x) \end{cases}$$

Find $w(x,t)$ along characteristics. $x = x(t)$. At this point we don't know the charact.

Define $\hat{W}(t) = W(x(t), t)$

$\Rightarrow \quad \dfrac{d\hat{w}}{dt} \overset{\text{chain rule}}{=} w_t + w_x \dfrac{dx}{dt}$

If $\boxed{\dfrac{dx}{dt} = t} \Rightarrow \dfrac{d\hat{w}}{dt} = w_t + t\,w_x \overset{\text{Equ.}}{=} 0.$

$\Rightarrow \quad \hat{w}(t) \equiv K$, when $x = x(t)$

Now, $x(t)$ satisfies

$$\frac{dx}{dt} = t \Rightarrow x(t) = \frac{t^2}{2} + C$$

when $\underline{t=0} \Rightarrow x(0) = C = x_0$

Then family of characteristics

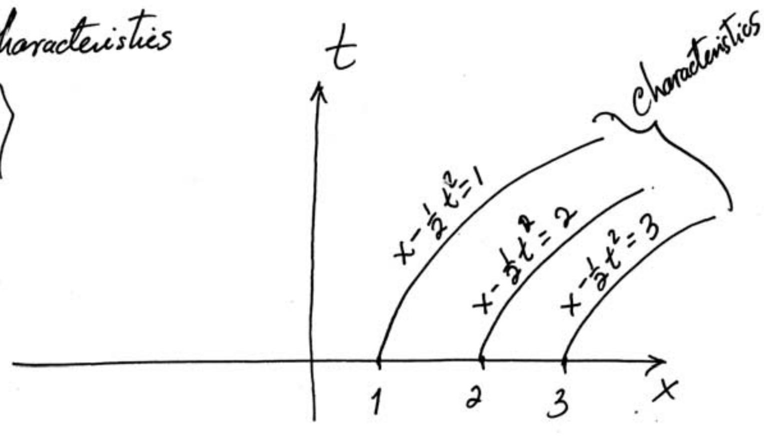
$$\boxed{x - \frac{1}{2}t^2 = x_0}$$

$x_0 = 1$

$x = \frac{1}{2}t^2 + 1$

$x_0 = 2$

$x = \frac{1}{2}t^2 + 2$



and the Soln. for the IVP along charact. $x = x(t)$

$$W(x,t) = W(x(t),t) = \hat{W}(t) = \text{Constant}$$

$$\Rightarrow \quad W(x(t),t) = W(x(0),0) = W(x_0,0) = f(x_0).$$

$\Rightarrow$ Along. Characteristic, $\quad w(x(t),t) = f(x_0) = f\left(x(t) - \frac{1}{2}t^2\right)$

For any (x,t) $\quad \boxed{w(x,t) = f\left(x - \frac{1}{2}t^2\right).}$

If $\begin{cases} w_t + tw_x = 1 \\ w(x,0) = f(x) \end{cases}$

Along characteristics: $\quad x = x(t).$

$$\hat{w}(t) = W(x(t),t)$$

$$\frac{d\hat{w}}{dt} = w_t + w_x \frac{dx}{dt} \overset{\boxed{\text{if } \frac{dx}{dt} = t}}{=} w_t + w_x t \overset{\text{equ.}}{=} 1$$

$\Rightarrow$ Along characteristics:

$$\boxed{\frac{dx}{dt}(t) = t} \Rightarrow x(t) = \frac{t^2}{2} + c \Rightarrow x - \frac{t^2}{2} = x_0$$

$$\boxed{\frac{d\hat{w}}{dt}(t) = 1} \Rightarrow \hat{w}(t) = t + k_1$$

Now, $\quad \hat{W}(0) = k_1 = W(x(0),0) = W(x_0,0) = f(x_0)$

$$\Rightarrow \quad \hat{W}(t) = t + f(x_0)$$

$$\Rightarrow \quad \boxed{W(x,t) = W(x(t),t) = \hat{w}(t) = t + f\left(x - t^2/2\right).}$$