Nonhomogeneous Problems (Sects. 3.4 and 8.2, 8.3).

Consider the following problems of Sect. 3.4

\# 3.4.9.

$$\begin{cases} u_t = k\, u_{xx} + q(x,t) \\ u(0,t) = 0, \quad u(L,t) = 0 \end{cases}$$

\# 3.4.11

$$\begin{cases} u_t = k\, u_{xx} + g(x) \\ u(0,t) = 0, \quad u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

\# 3.4.12

$$\begin{cases} u_t = k\, u_{xx} + e^{-t} + e^{-2t} \cos \frac{3\pi}{L} x \\ u_x(0,t) = 0, \quad u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

\# 3.4.13

$$\begin{cases} u_t = k\, u_{xx} \\ u(0,t) = A(t), \quad u(L,t) = 0 \\ u(x,0) = g(x) \end{cases}$$

\# 8.3.7 Solve the IBVP

$$\begin{cases} u_t = u_{xx}, & 0<x<L, \quad t>0 \quad (1) \\ u(0,t)=0 & (2) \\ u(L,t)=t & (3) \\ u(x,0)=0 & (4) \end{cases}$$

Because B.C. depends on time <u>there is not an equilibrium Solution.</u>

But, we still can introduce a change of dependent variables that will transform our original homogeneous equation into a nonhomogeneous one, but the B.C. will be homogeneous. In fact, defining

$$\boxed{r(x,t) \equiv A(t) + \frac{(B(t)-A(t))}{L}x = \frac{t}{L}x}$$

and $\boxed{v(x,t) \equiv u(x,t) - r(x,t)}$ $\Rightarrow u(x,t) = v(x,t) + r(x,t)$

$$\Rightarrow \quad 0 = u_t - u_{xx} = v_t + r_t - v_{xx} - r_{xx} =$$

$$= v_t + \frac{x}{L} - v_{xx} - 0$$

$$\Rightarrow \quad \boxed{v_t = v_{xx} - \frac{x}{L}} = Q(x)$$

The B.C's transform into homogeneous B.C's. for $v(x,t)$.

$$v(0,t) = u(0,t) - r(0,t) = 0 - 0 = 0$$

$$v(L,t) = u(L,t) - r(L,t) = t - t = 0$$

and the I.C. transform into

$$v(x,0) = u(x,0) + r(x,0) = 0 + 0 = 0.$$

Summarizing, the new IBVP for the new dependent variable $v(x,t)$ is given by

$$\begin{cases} v_t = v_{xx} - \dfrac{x}{L} & (5) \\ v(0,t) = 0, \quad v(L,t) = 0 & (6) \\ v(x,0) = 0 & (7) \end{cases}$$

this new problem <u>admits an equilibrium solution $v_E(x)$</u>

In fact,

$$\begin{cases} v_E''(x) = \dfrac{x}{L} \\ v_E(0) = 0, \quad v_E(L) = 0 \end{cases}$$

$$\Rightarrow v_E'(x) = \frac{x^2}{2L} + C_1 \Rightarrow v_E(x) = \frac{x^3}{6L} + C_1 x + C_2$$

Using the B.C's. $0 = v_E(0) = C_2 \Rightarrow C_2 = 0$

and $0 = v_E(L) = \frac{\cancel{L^3}\,L^2}{6\cancel{L}} + C_1 L \Rightarrow C_1 = -\frac{L}{6}$

$$\Rightarrow \boxed{v_E(x) = \frac{x^3}{6L} - \frac{L}{6}x}$$

Therefore, if we define

$$\boxed{\hat{v}(x,t) = v(x,t) - v_E(x)}, \quad \text{or} \quad v(x,t) = \hat{v}(x,t) + v_E(x)$$

The new dependent variable $\hat{v}(x,t)$ will satisfy

$$0 = v_t - v_{xx} + \frac{x}{L} = \hat{v}_t + 0 - \hat{v}_{xx} - v_E''(x) + \frac{x}{L}$$

$$= \hat{v}_t - \hat{v}_{xx} - \frac{x}{L} + \frac{x}{L}$$

$$\Rightarrow \begin{cases} \hat{v}_t = \hat{v}_{xx} \\ \hat{v}(0,t) = \overset{=0}{v(0,t)} - \overset{=0}{v_E(0)} = 0 \\ \hat{v}(L,t) = v(L,t) - v_E(L) = 0 - 0 = 0 \\ \hat{v}(x,0) = v(x,0) - v_E(x) = -v_E(x) = \frac{L}{6}x - \frac{x^3}{6L} \end{cases}$$

whose solution is given (after sep. of variables) by

$$\hat{v}(x,t) = \sum_{n=1}^{\infty} \hat{a}_n \sin \frac{n\pi}{L}x \, e^{-\left(\frac{n\pi}{L}\right)^2 t} \quad \text{where} \quad \boxed{\hat{a}_n \equiv \frac{2}{L}\int_0^L \left[\frac{L}{6}x - \frac{x^3}{6L}\right] \sin \frac{n\pi}{L}x \, dx}$$

$$\Rightarrow \boxed{v(x,t) = \hat{v}(x,t) + v_E(x) = \sum_{n=1}^{\infty} \hat{a}_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} + \frac{x^3}{6L} - \frac{L}{6}x}$$

and finally

$$\boxed{u(x,t) = v(x,t) + r(x,t) = \sum_{n=1}^{\infty} \hat{a}_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 t} + \frac{x^3}{6L} - \frac{L}{6}x + \frac{t}{L}x}$$