Qualitative Properties of Laplace's Equation.

Consider the BVP:

$$\begin{cases} \nabla^2 u = 0, & \vec{x} \in \Omega \quad (1) \\ u(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega \quad (2) \end{cases}$$

where $h(\vec{x}_s)$ is conts. on $\partial\Omega$.

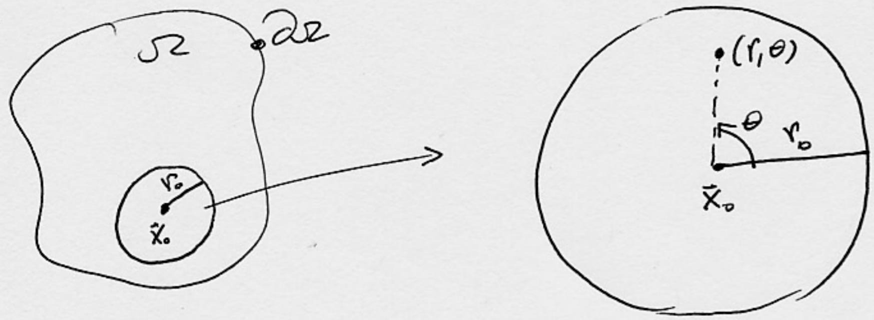
Concept of well-posedenes

If BVP (1)-(2) has a unique solution that depends continuously on the boundary conditions, the BVP is said to be well-posed.

Thm (Mean value theorem)

The value of $u(\vec{x}_0)$ soln. of (1)-(2) at any interior point $\vec{x}_0$ is the average of u along any circle of radius r_0 centered centered at $\vec{x}_0$.

Proof: For any circular disk $D(\vec{x}_0; r_0)$ bounded by a circle centered at $\vec{x}_0$ of radius r_0 contained in Ω, we can define a polar coordinate system with center at $\vec{x}_0$.



$$r \equiv \sqrt{(x-x_0)^2+(y-y_0)^2}$$

$$\tan\theta = \frac{y-y_0}{x-x_0}$$

From our previous result for a circular disk,

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos(n\theta)+B_n\sin(n\theta)\right]r^n, \quad \text{in } D(\vec{x}_0,r_0)$$

In particular, at $r=0$

$$u(\vec{x}_0) = u(0,\theta) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r_0,\theta)\,d\theta = \text{Average along circle of radius } r_0.$$

Thm (Max pple)

The maximum and minimum of u soln. of (1)-(2) occur on the boundary $\partial\Omega$.

Proof.- There are two cases:

i) The soln. u is constant everywhere including the boundaries. In this case, the max. pple is trivially true.

ii) The soln u is not constant everywhere

If in this case, we assume that the max value of u occurs in the interior of Ω at a point $\vec{x} = \vec{x}_0$.

Let's construct a circle $B_{r_0}(\vec{x}_0) \subset \Omega$. Then, according to the mean value theorem,

$$u(\vec{x}_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{u(r_0, \theta)}_{\substack{\| \\ u(\vec{x}_c)}} \, d\theta \,, \qquad \vec{x}_c \in B_{r_0}(\vec{x}_0)$$

That means that it should be at least one point on $B_{r_0}(\vec{x}_c)$ where $u(\vec{x}_0) < u(\vec{x}_c)$ contradicting the hypothesis.

If not $u(\vec{x}_c) = u(\vec{x}_0)$, for all $\vec{x}_c \in B_{r_0}(\vec{x}_0)$.

This leads to a constant solution or case (i).

Thm.- The solution of BVP (1)-(2) satisfies:

i) If its solution exists then it's unique.

ii) " " " " " it depends continuously on the boundary cond.

Proof.- i) Assume $u_1(\vec{x})$ and $u_2(\vec{x})$ are two solutions of (1) satisfying (2), then

$$w(x) \equiv u_1(x) - u_2(x)$$

Satisfies (1) and $w(\vec{x}_s) = u_1(\vec{x}_s) - u_2(\vec{x}_s) = h(\vec{x}_s) - h(\vec{x}_s) \equiv 0$

for all $\vec{x}_s \in \partial\Omega$.

Using max pple

$$0 = \min_{\vec{x}_s \in \partial\Omega} (w(\vec{x}_s)) \le w(x) \le \max_{\vec{x}_s \in \partial\Omega} (w(\vec{x}_s)) = 0$$

$\Rightarrow \quad w(\vec{x}) \equiv 0, \quad \vec{x} \in \Omega \cup \partial\Omega \Rightarrow u_1(\vec{x}) = u_2(\vec{x}),$

for all $\vec{x} \in \Omega \cup \partial\Omega$.

ii) Assume $u(x)$ is a soln. of (1)-(2)

$w(x)$ is a soln. of (1) with B.C. $u(\vec{x}_s) = g(\vec{x}_s)$

$\vec{x}_s \in \partial\Omega$.

and $|h(\vec{x}_s) - g(\vec{x}_s)| < \varepsilon, \quad \varepsilon > 0.$

then

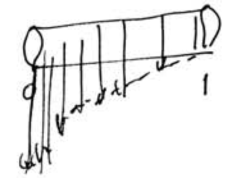
$$-\varepsilon < \min_{\vec{x}_s \in \partial\Omega} (h(\vec{x}_s) - g(\vec{x}_s)) \le u(\vec{x}) - w(\vec{x}) \le \max_{\vec{x}_s \in \partial\Omega} (h(\vec{x}_s) - g(\vec{x}_s)) < \varepsilon$$

$$\Rightarrow \quad |u(\vec{x}) - w(\vec{x})| < \varepsilon, \quad \vec{x} \in \Omega \cup \partial\Omega.$$

2.5 Solvability condition

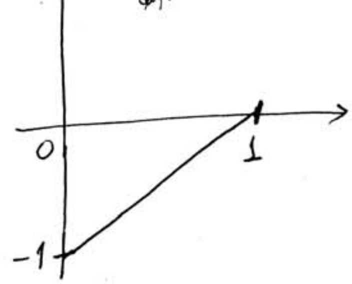
Problem 1.4.7 with $\beta = 1$, $L = 1$

$$\begin{cases} u_t = u_{xx} + (x-1) \\ u(x,0) = f(x) \\ u_x(0,t) = 0, \quad u_x(1,t) = 0 \end{cases}$$



Equilibrium problem:

$$\begin{cases} \hat{u}'' = -x+1 \\ \hat{u}'(0) = 0, \quad u'(1) = 0 \end{cases}$$



$$\Rightarrow \quad \hat{u}'(x) = -\frac{x^2}{2} + x + C_1$$

Using $\underline{\underline{B.C.}}$ $\quad \hat{u}'(0) = 0 \Rightarrow \boxed{C_1 = 0}$

$\hat{u}'(1) = 0 \Rightarrow -\frac{1}{2} + 1 = 0$ Contradiction!

There is not an equilibrium.

We also notice that

$$\int_0^1 \hat{u}''(x) = \hat{u}'(x)\Big|_0^1 = \hat{u}'(1) - \hat{u}'(0) = 0$$

and $$\int_0^1 \hat{u}''(x)\,dx = \int_0^1 (-x+1)\,dx = -\frac{x^2}{2} + x\Big|_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}.$$

Therefore, for the IBVP:

$$\begin{cases} u_t = K u_{xx} + Q(x) \\ u(x,0) = f(x) \\ u_x(0,t) = A, \qquad u_x(L,t) = B \end{cases}$$

An equilibrium or solution for the BVP exists

$$\begin{cases} \hat{u}''(x) = -\dfrac{Q(x)}{K} \\ \hat{u}'(0) = 0, \qquad \hat{u}'(L) = 0 \end{cases}$$

if $$\int_0^L \hat{u}''(x)\,dx = \int_0^L -\frac{Q(x)}{K}\,dx$$

$$\Rightarrow \quad \hat{u}'(L) - \hat{u}'(0) = -\int_0^L \frac{Q(x)}{K}\,dx$$

or $$\boxed{B - A = -\int_0^L \frac{Q(x)}{K}\,dx.}$$

This condition is usually called Solvability or Compatibility condition.

In 2-D or 3-D, the equilibrium problem for a Neumann problem is given by

$$\begin{cases} \nabla^2 u = \dfrac{-Q(\vec{x})}{K}, & \vec{x} \in \Omega \\ \dfrac{\partial u}{\partial n}(\vec{x}_s) = f(\vec{x}_s), & \vec{x}_s \in \partial\Omega. \end{cases}$$

Therefore, $$\iiint_{\Omega} \nabla^2 u(\vec{x})\, dv = -\iiint_{\Omega} \frac{Q(\vec{x})}{K}\, dv$$

But also, $$\iiint_{\Omega} \nabla^2 u(\vec{x})\, dv \overset{\text{Gauss Thm}}{=} \iint_{\partial\Omega} \nabla u \cdot \hat{n}\, ds$$

Then, a necessary cond. for an equilibrium solution to exists is that

$$\boxed{\iint_{\partial\Omega} (\nabla u \cdot \hat{n})(\vec{x}_s)\, ds = -\iiint_{\Omega} \frac{Q(\vec{x})}{K}\, dv}$$

Compatibility condition.

In the case of a Neumann equation a BVP as

$$\begin{cases} \nabla^2 u = 0, & \vec{x} \in \Omega \\ \dfrac{\partial u}{\partial n}(\vec{x}_s) = h(\vec{x}_s), & \vec{x}_s \in \partial\Omega \end{cases}$$

a necessary condition for a solution to exist is that

$$0 = \iiint_{\Omega} \nabla^2 u(\vec{x})\, dv \overset{\text{Gauss Thm}}{=} \iint_{\partial\Omega} \frac{\partial u}{\partial n}(\vec{x}_s)\, ds = \iint_{\partial\Omega} h(\vec{x}_s)\, ds$$

means

$$\boxed{\iint_{\partial\Omega} \nabla u \cdot \hat{n}\, ds = \iint_{\partial\Omega} h(\vec{x}_s)\, ds = 0.}$$

Solution for Laplace's equation inside a circular disk.

$$\begin{cases} \nabla^2_{r,\theta} u = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 & (1) \\ u(a,\theta) = f(\theta) & (2) \end{cases}$$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta)\, r^n + \sum_{n=1}^{\infty} B_n \sin(n\theta)\, r^n \qquad (3)$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \qquad A_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta \qquad (4)$$

$$B_n a^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta \qquad (5)$$

Subst. (4) - (5) into (3).

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta + \sum_{n=1}^{\infty} \frac{1}{\pi}\left(\int_{-\pi}^{\pi} f(\bar\theta)\cos(n\bar\theta)\,d\bar\theta\right)\cos(n\theta)\,\frac{r^n}{a^n}$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} f(\bar\theta)\sin(n\bar\theta)\,d\bar\theta\right)\sin(n\theta)\,\frac{r^n}{a^n}$$

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\bar\theta)\,d\bar\theta + \sum_{n=1}^{\infty}\frac{1}{\pi}\left[\left(\int_{-\pi}^{\pi} f(\bar\theta)\cos n\bar\theta\,d\bar\theta\right)\frac{r^n}{a^n}\cos n\theta + \right.$$

$$\left. + \sum_{n=1}^{\infty}\frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(\bar\theta)\sin n\bar\theta\,d\bar\theta\right]\frac{r^n}{a^n}\sin n\theta\right] =$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{\pi} f(\bar\theta)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left[\cos n\bar\theta\cos n\theta + \sin n\bar\theta\sin\theta\right]\right]d\bar\theta\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{\pi} f(\bar\theta)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n\theta - n\bar\theta)\right]d\bar\theta\right\}$$

$\therefore$

$$\boxed{u(r,\theta) = \frac{1}{\pi}\left\{\int_{-\pi}^{\pi} f(\bar\theta)\left[-\frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\cos(n\theta - n\bar\theta)\right]d\bar\theta\right\}} \quad (1.1)$$

Using $\cos z = \mathrm{Re}\left[e^{iz}\right]$ sum the resulting geometric series to obtain Poisson's integral formula.

$$\boxed{u(r,\theta) = \frac{1}{2\pi}\int_{0}^{2\pi} f(\theta_0)\,\frac{a^2 - r^2}{r^2 + a^2 - 2ar\cos(\theta - \theta_0)}\,d\theta_0} \quad (1.2)$$