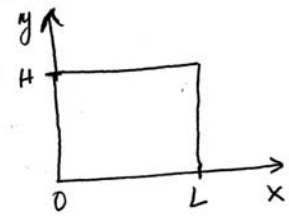
7.3 Vibrating Rectangular Membrane.

IBVP:

$$u_{tt} = c^2 (u_{xx} + u_{yy}) = c^2 \nabla_{xy}^2 u \qquad (1)$$



BC's
$$\begin{cases} u(0,y,t) = 0, \quad u(x,0,t) = 0 & (2) \\ u(L,y,t) = 0, \quad u(x,H,t) = 0 & (3) \end{cases}$$

IC's
$$\begin{cases} u(x,y,0) = \alpha(x,y) & (4) \\ u_t(x,y,0) = \beta(x,y) & (5) \end{cases}$$

Separation of variables:

$$\boxed{u(x,y,t) = h(t)\,\phi(x,y)}$$

Substitution in (1) leads to

$$h''\phi = c^2 h\,(\phi_{xx} + \phi_{yy}) \xRightarrow[\text{by } c^2 h\phi]{\text{Dividing}} \frac{h''}{c^2 h} = \frac{1}{\phi}(\phi_{xx} + \phi_{yy}) = -\lambda$$

Explain why?

Therefore, we obtain the two equations (ODE):

$$\begin{cases} h'' - c^2 \lambda h = 0 \\ \phi_{xx} + \phi_{yy} = -\lambda \phi \end{cases}$$

From the boundary conditions:

$\phi(0,y)\,h(t) = 0 \Rightarrow \phi(0,y) = 0$ | $\phi(x,0)\,h(t) = 0 \Rightarrow \phi(x,0) = 0$

$\phi(L,y)\,h(t) = 0 \Rightarrow \phi(L,y) = 0$ | $\phi(x,H)\,h(t) = 0 \Rightarrow \phi(x,H) = 0$

Two-dimensional eigenvalue problem:

$$\begin{cases} \nabla^2_{x,y}\phi = \phi_{xx} + \phi_{yy} = -\lambda\phi & (2.1) \\ \phi(0,y) = 0, \quad \phi(x,0) = 0 & (2.2) \\ \phi(L,y) = 0, \quad \phi(x,H) = 0 & (2.3) \end{cases}$$

The equation for $h(t)$ has been solved for the 1-D problem

$$h''(t) + \lambda c\, h(t) = 0$$

Assuming $\lambda > 0$ (oscilatory solutions).

$$\boxed{h(t) = C_1 \cos c\sqrt{\lambda}t + C_2 \sin c\sqrt{\lambda}t} \qquad (2.4)$$

The eigenvalue problem is another PDE with homogeneous boundary conditions (2.2) – (2.3).

We try a further separation of variables in the variables x, y

$$\phi(x,y) = f(x)\, g(y) \qquad (2.5)$$

Substitution of (2.5) into (2.1) leads to

$$g(y)\,f''(x) + f(x)\,g''(y) = -\lambda\, f(x)\,g(y)$$

Dividing by $f(x)g(y)$.

$$\frac{f''(x)}{f} = -\lambda - \frac{g''(y)}{g(y)} = -\mu$$

Two ODE's result from this equation:

$$\boxed{f'' + \mu f = 0} \quad \text{and} \quad \boxed{g'' + (\lambda - \mu)\,g = 0}$$

And using the BC's (2.2) and (2.3), we obtain two eigenvalue problems:

$$\boxed{\begin{array}{l} f'' + \mu f = 0 \\ f(0) = 0, \quad f(L) = 0 \end{array}} \qquad (3.1)$$

and

$$\boxed{\begin{cases} g'' + (\lambda - \mu)\,g = 0 \\ g(0) = 0, \quad g(H) = 0 \end{cases}} \qquad (3.2)$$

Eigenvalues for (3.1) are $\boxed{\mu_n = \left(\frac{n\pi}{L}\right)^2}$, $n = 1, 2, \ldots$ (3.3)

and eigenfunctions are $\boxed{f_n(x) = \sin\left(\frac{n\pi}{L}x\right).}$ (3.4)

For each $n = 1, 2, \ldots$ (3.2) defines an eigenvalue problem:

$$\begin{cases} g_n'' + (\lambda - \mu_n) g_n = 0 \\ g_n(0) = 0, \quad g_n(H) = 0 \end{cases}$$

whose eigenvalues are $\boxed{\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2}$, $m = 1, 2, \ldots$ (4.1)

and corresponding eigenfunctions are

$$\boxed{g_{nm}(y) = \sin\left(\frac{m\pi}{H} y\right)} \quad (4.2)$$

Combining (3.3) and (4.1)

$$\boxed{\lambda_{nm} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2} \quad \begin{matrix} n = 1, 2, \ldots \\ m = 1, 2, \ldots \end{matrix}$$

and we have obtained the eigenvalues of the EVP. (2.1)-(2.3).

From (2.5), (3.4) and (4.2), we also obtain the eigenfunctions:

$$\boxed{\phi_{nm} = \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{H} y\right)} \quad \begin{matrix} n = 1, 2, \ldots \\ m = 1, 2, \ldots \end{matrix}$$

Principle of Superposition:

Two families of product solutions combine to form the soln:

$$\boxed{\begin{aligned} u(x,t) = & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right) \cos\left(c\sqrt{\lambda_{nm}}\, t\right) \\ & + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{nm} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right) \sin\left(c\sqrt{\lambda_{nm}}\, t\right) \end{aligned}} \tag{5.1}$$

The coefficients A_{nm} and B_{nm} can be determined from the IC's

$$\boxed{\overset{\text{Known}}{\alpha}(x,y) = u(x,y,0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right)}$$

Double Fourier Series.

There is a theory for this double F.S. similar to single F.S.

However, it can be treated as two iterated single F.S.

We will assume that $\alpha(x,y)$ and $\beta(x,y)$ are at least piecewise smooth in each variable.

In fact,

$$\alpha(x,y) = \sum_{m=1}^{\infty} \underbrace{\left(\sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right)\right)}_{C_m(x)} \sin\left(\frac{m\pi}{H}y\right) \quad (6.1)$$

For x fixed,

$$\boxed{C_m(x) = \sum_{n=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{L}x\right) = \underbrace{\frac{2}{H}\int_0^H \alpha(x,y)\sin\left(\frac{m\pi}{H}y\right)dy}_{\equiv\ K(x).}} \quad (6.2)$$

The equation (6.2) is the Fourier Sine Series of $K(x)$

then, $$A_{nm} = \frac{2}{L}\int_0^L K(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

or

$$\boxed{\begin{aligned} A_{nm} &= \frac{2}{L}\int_0^L \left[\frac{2}{H}\int_0^H \alpha(x,y)\sin\left(\frac{m\pi}{H}y\right)dy\right] \sin\frac{n\pi}{L}x \; dx \\ &= \frac{4}{LH}\underbrace{\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{m\pi}{H}y\right)\sin\left(\frac{n\pi}{L}x\right)dy\,dx.}_{\text{Double Integral}} \end{aligned}} \quad (6.3)$$

$n = 1, 2, \ldots$

$m = 1, 2, \ldots$

From $u_t(x,y,0) = \beta(x,y)$

$$u_t(x,y,0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{nm} c\sqrt{\lambda_{nm}} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{H}y\right) = \beta(x,y)$$

$$\Rightarrow \boxed{c\sqrt{\lambda_{nm}}\, B_{nm} = \frac{4}{LH} \int_0^L \int_0^H \beta(x,y) \sin\left(\frac{m\pi}{H}y\right) \sin\left(\frac{n\pi}{L}x\right) dy\, dx} \quad (7.1)$$

$m = 1, 2, \ldots$
$n = 1, 2, \ldots$

Summarizing, The solution of (1)-(5) is given by (5.1) with the coefficients A_{nm} and B_{nm} defined by (6.3) and (7.1).

All this separation process

$$u(x,y,t) = \phi(x,y)\, h(t)$$

and later $\phi(x,y) = f(x)\, g(y)$

is equivalent to start from the beginning with

$$u(x,y,t) = f(x)\, g(y)\, h(t).$$

MODES OF VIBRATIONS.

Consider the IBVP.

$$\begin{cases} u_{tt} = c^2 \nabla^2_{x,y} u \\ u(0,y,t) = 0, \quad u(x,0,t) = 0 \\ u(L,y,t) = 0, \quad u(x,H,t) = 0 \\ u(x,y,0) = \sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{H}y\right). \\ u_t(x,y,0) = 0 \end{cases}$$

Soln:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{nm}\cos\left(c\sqrt{\lambda_{nm}}\,t\right) + B_{nm}\sin\left(c\sqrt{\lambda_{nm}}\,t\right)\right] *$$

$$* \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)$$

Using I.C's:

$$u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right) = \sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{H}y\right)$$

Using orthogonality: $A_{11} = 1$, $A_{nm} = 0$, all other case.

From the 2nd I.C. $u_t(x,y,0) = 0 \Rightarrow B_{nm} = 0$, for all n,m.

$\therefore$

$$\boxed{u(x,y,t) = \cos\left(c\sqrt{\lambda_{11}}\,t\right)\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{H}y\right)}$$

The previous Soln. Constitutes the most elementary mode of Vibration. All other modes Can be obtained

1) By defining initial conditions:

$$u(x,y,0) = \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right). \quad \begin{array}{l} n=1,2,\dots \\ m=1,2,\dots \end{array}$$

$$u_t(x,y,0) = 0$$

or

2) $$u(x,y,0) = 0$$

$$u_t(x,y,0) = \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right).$$