12.3 Method of Characteristics for the One-Dimensional Wave equation.

$$\boxed{u_{tt} = c^2 u_{xx}} \quad t>0, \ -\infty<x<\infty \quad (1)$$

We want to show that the "General Solution" for this equation can be expressed in terms of two unknown functions: $F(x)$ and $G(x)$ as

$$u(x,t) = F(x-ct) + G(x+ct).$$

Let's start with the 1^{st} order wave equation:

$$u_t + c\, u_x = 0, \quad t>0, \quad -\infty<x<\infty$$

We have shown that its general solution can be written in terms of an unknown function: $F(x)$.

as $\quad u(x,t) = F(x-ct).$

An alternative derivation of this result is as follows:

Introduce the change of Variables:

$$\xi = \xi(x,t) = x - ct,$$

And by inverting the above formula:

$$x = X(\xi,t) = \xi + ct,$$

Define: $\hat{u}(\xi,t) = u(x(\xi,t),t)$

$$\Rightarrow \quad u(x,t) = \hat{u}(\xi(x,t),t)$$

Thus, $0 = u_t + c u_x = \hat{u}_\xi \xi_t + \hat{u}_t + c\,\hat{u}_\xi \xi_x$

$$= \cancel{-c\hat{u}_\xi} + \hat{u}_t + \cancel{c\,\hat{u}_\xi(1)} = \hat{u}_t$$

$$\Rightarrow \quad \hat{u}_t(\xi,t) = 0 \Rightarrow \hat{u}(\xi,t) = F(\xi)$$

$$\Rightarrow \quad u(x,t) = \hat{u}(\xi(x,t),t) = F(\xi(x,t)) = F(x-ct).$$

or $\boxed{u(x,t) = F(x-ct)}$

Back to equ (1) 2^{nd} order wave equation.

Two change of variables: $\xi = \xi(x,t) = x - ct$
$\eta = \eta(x,t) = x + ct$

Define: $\hat{u}(\xi,\eta) = u(x(\xi,\eta), t(\xi,\eta))$

inverting the change

$$\Rightarrow u(x,t) = \hat{u}(\xi(x,t), \eta(x,t))$$

$$\Rightarrow u_t = \hat{u}_\xi \xi_t + \hat{u}_\eta \eta_t, \quad u_x = \hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x$$

$$\Rightarrow u_t = \hat{u}_\xi(-c) + \hat{u}_\eta(c), \quad \Rightarrow u_x = \hat{u}_\xi(1) + \hat{u}_\eta(1).$$

$$\Rightarrow u_{tt} = (-c)[\hat{u}_{\xi\xi}\xi_t + \hat{u}_{\xi\eta}\eta_t] + c[\hat{u}_{\eta\xi}\xi_t + \hat{u}_{\eta\eta}\eta_t] = c^2\hat{u}_{\xi\xi} - c^2 u_{\xi\eta} - c^2 u_{\eta\xi} + c^2 u_{\eta\eta}$$

So

$$u_{tt} = c^2 \hat{u}_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}$$

and

$$u_{xx} = \frac{\partial}{\partial x}\left[\hat{u}_\xi + \hat{u}_\eta\right] = \hat{u}_{\xi\xi}\overset{=1}{\xi_x} + \hat{u}_{\xi\eta}\overset{=1}{\eta_x} + \hat{u}_{\eta\xi}\overset{=1}{\xi_x} + \hat{u}_{\eta\eta}\overset{=1}{\eta_x}$$

$$\Rightarrow \quad u_{xx} = \hat{u}_{\xi\xi} + 2\hat{u}_{\xi\eta} + \hat{u}_{\eta\eta}$$

Substitution into (1)

$$0 = u_{tt} - c^2 u_{xx} = c^2\left[\cancel{\hat{u}_{\xi\xi}} - 2\hat{u}_{\xi\eta} + \cancel{u_{\eta\eta}}\right] - c^2\left[\cancel{\hat{u}_{\xi\xi}} + 2\hat{u}_{\xi\eta} + \cancel{\hat{u}_{\eta\eta}}\right]$$

$$= -4c^2 \hat{u}_{\xi\eta}$$

$$\Rightarrow \quad \boxed{\hat{u}_{\xi\eta} = 0}$$

Integrating w.r.t. η

$$\hat{u}_\xi(\xi,\eta) = f(\xi)$$

Integrating w.r.t. ξ

$$\hat{u}(\xi,\eta) = \int f(\xi)\,d\xi + G(\eta) = F(\xi) + G(\eta).$$

$$\text{or} \quad \boxed{\hat{u}(\xi,\eta) = F(\xi) + G(\eta)}$$

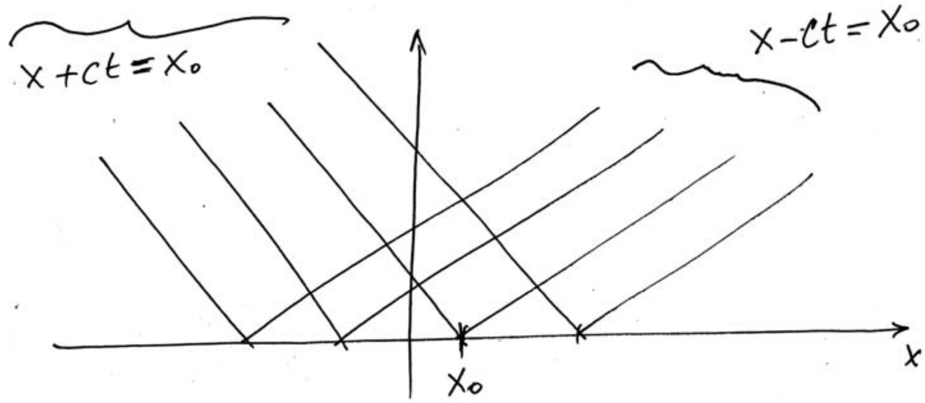
Therefore,

$$u(x,t) = \hat{u}\left(\xi(x,t), \eta(x,t)\right) = F\left(\xi(x,t)\right) + G\left(\eta(x,t)\right)$$

or

$$\boxed{u(x,t) = F(x-ct) + G(x+ct)}$$ ✓

Two families of characteristics.



IVP (Infinite domain).

Thm.- The IVP.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t>0, \quad -\infty<x<\infty \\ u(x,0) = f(x), & \\ u_t(x,0) = g(x), & -\infty<x<\infty \end{cases}$$

has a unique Sdn. and it can be represented as

$$u(x,t) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} \qquad (4.1)$$

where $f(x)$ is twice differentiable and $g(x)$ is differentiable.

Summarizing pages ①-④.

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Define:

$$\xi = \xi(x,t) = x - ct, \qquad \eta = \eta(x,t) = x + ct.$$

Inverse: $x = x(\xi,\eta) = \frac{\xi+\eta}{2}, \qquad t = t(\xi,\eta) = \frac{\eta - \xi}{2c}$

Define: $\hat{u}(\xi,\eta) = u(x(\xi,\eta), t(\xi,\eta))$

Inverse: $u(x,t) = \hat{u}(\xi(x,t), \eta(x,t))$

We prove that

$$u_{tt} - c^2 u_{xx} = 0 \Rightarrow \hat{u}_{\xi\eta}(\xi,\eta) = 0.$$

Integrating once w.r.t. η

$$\hat{u}_\xi(\xi,\eta) = f(\xi)$$

Integrating now w.r.t. ξ

$$\hat{u}(\xi,\eta) = \int f(\xi)\,d\xi + G(\eta) = F(\xi) + G(\eta).$$

$$\Rightarrow u(x,t) = \hat{u}(\xi(x,t), \eta(x,t)) = F(\xi(x,t)) + G(\eta(x,t)).$$

or $\boxed{u(x,t) = F(x-ct) + G(x+ct)}$ ✓

First, we will verify that (4.1) is a solution of the wave equ.

$$u_t \overset{\text{Leibniz's rule}}{=} \frac{1}{2}\left[f'(x-ct)(-c) + f'(x+ct)(c)\right] + \frac{1}{2c}\left[g(x+ct)*c - g(x-ct)*(-c)\right]$$

$$\Rightarrow u_{tt} = \frac{1}{2}\left[c^2 f''() + c^2 f''\right] + \frac{1}{2c}\left[c^2 g'() - c^2 g'()\right]$$

Similarly,

$$u_{xx} = \frac{1}{2}\left[f''() + f''() + \frac{1}{c}\left(g'() - g'()\right)\right]$$

Thus,

$$u_{tt} - c^2 u_{xx} = 0 \checkmark$$ and (4.1) is a soln. of the wave equation if $f(x)$ is twice differentiable and $g(x)$ has 1st derivative.

Eq. (4.1) also satisfies the I.C's., since

$$u(x,0) = \frac{f(x)+f(x)}{2} + \frac{1}{2c}\int_x^x \overset{0}{\cancel{g(\bar{x})\,dx}} = f(x) \checkmark$$

$$u_t(x,t) = \frac{1}{2}\left[-cf'(x-ct) + cf'(x+ct)\right] + \frac{1}{2c}\left[g(x+ct)c + c\,g(x-ct)\right]$$

$$\Rightarrow u_t(x,0) = \frac{1}{2}\left[\overset{0}{\cancel{-cf'(x) + cf'(x)}}\right] + \frac{1}{2c}\left[cg(x) + cg(x)\right] = g(x) \checkmark$$

Leibniz rule:

$$\frac{d}{dx}\left[\int_{f(x)}^{g(x)} h(t,x)\,dt\right] = \int_{f(x)}^{g(x)} \frac{\partial h}{\partial x}(t,x)\,dt + h(t,g(x))\,g'(x) - h(t,f(x))\,f'(x).$$

Uniqueness:

All Solutions of the wave equation are of the form:

$$u(x,t) = F(x-ct) + G(x+ct) \quad (6.0)$$ F, G arbitrary Twice differentiable.

We will prove that

$$F(x) = \frac{1}{2}\left[f(x) - \frac{1}{c}\int_a^x g(\bar{x})\,d\bar{x} \right] + K \qquad (6.1)$$

and

$$G(x) = \frac{1}{2}\left[f(x) + \frac{1}{c}\int_a^x g(\bar{x})\,d\bar{x} \right] - K. \qquad (6.2)$$

Therefore, $u(x,t) = F(x-ct) + G(x+ct) = \overset{\text{Equ.}}{(4.1)}$. ✓

To obtain (6.1) and (6.2) from (6.0), we use the IC's.

$$f(x) = u(x,0) = F(x) + G(x). \qquad (6.3)$$

$$g(x) = u_t(x,0) = -cF'(x) + cG'(x). \qquad (6.4)$$

Taking derivative of (6.3), multiplying it by (-c) and adding to (6.4)

$$-cf'(x) = -cF'(x) - \cancel{cG'(x)}$$

$$g(x) = -cF'(x) + \cancel{cG'(x)}$$

$$-cf'(x) + g(x) = -2cF'(x)$$

Integrating $\int_a^x dx$

$$cF(x) = \frac{1}{2}\left[cf(x) - \int_a^x g(\bar{x})\,d\bar{x} \right]$$

or $$\boxed{F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_a^x g(\bar{x})\, d\bar{x} + K.}$$

Substitution into (6.3)

$$G(x) = f(x) - F(x) = f(x) - \frac{1}{2} f(x) + \frac{1}{c} \int_a^x g(\bar{x})\, d\bar{x} - K$$

$$\Rightarrow \quad \boxed{G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_a^x g(\bar{x})\, d\bar{x} - K}$$

Therefore,

$$u(x,t) = F(x-ct) + G(x+ct) =$$

$$= \frac{1}{2} f(x-ct) - \frac{1}{c} \int_a^{x-ct} g(\bar{x})\, d\bar{x} + \cancel{K}$$

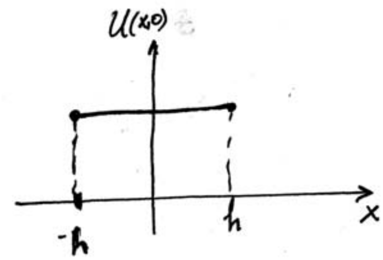
$$+ \frac{1}{2} f(x+ct) + \frac{1}{c} \int_a^{x+ct} g(\bar{x})\, d\bar{x} - \cancel{K}$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right] + \frac{1}{c} \int_{x-ct}^{x+ct} g(\bar{x})\, d\bar{x}} \qquad (7.1)$$

D'Alembert formula.

Example:- Solve the IVP.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\infty < x < \infty, \quad t > 0. \\ u(x,0) = f(x) = \begin{cases} 1, & |x| < h \\ 0, & |x| > h \end{cases} \\ u_t(x,0) = g(x) \equiv 0. \end{cases}$$



D'Alambert's Soln.

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right]$$

In Region I: $-h < x - ct < h$ and $-h < x + ct < h$

$$u(x,t) = \frac{1}{2}\left[1 + 1 \right] = 1.$$

In Region II: $-h < x-ct < h$ and $x+ct > h$.

$$u(x,t) = \frac{1}{2}\left[\overset{=1}{f(x-ct)} + \overset{=0}{f(x+ct)} \right] = \frac{1}{2}.$$

In Region III: $x-ct < -h$ and $|x+ct| < h$.

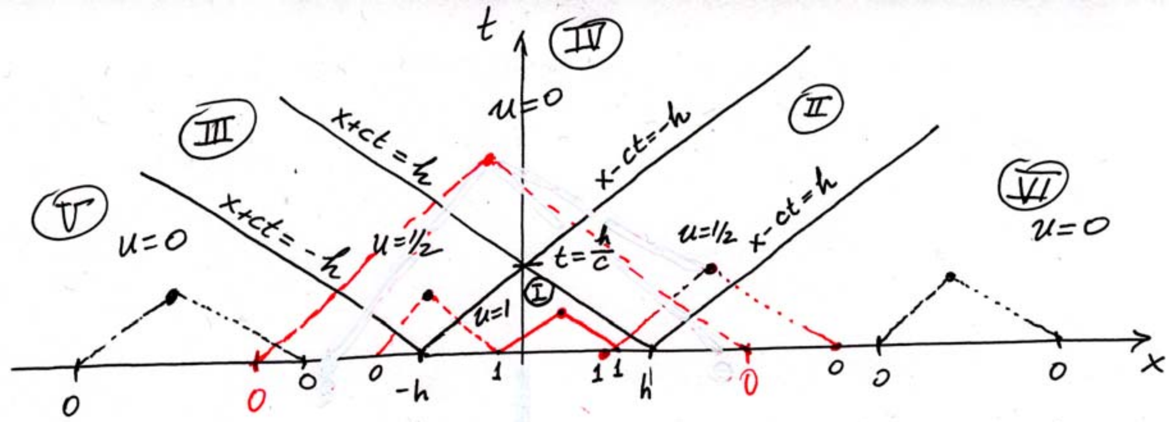
$$u(x,t) = \frac{1}{2}\left[\overset{=0}{f(x-ct)} + \overset{=1}{f(x+ct)} \right] = \frac{1}{2}.$$

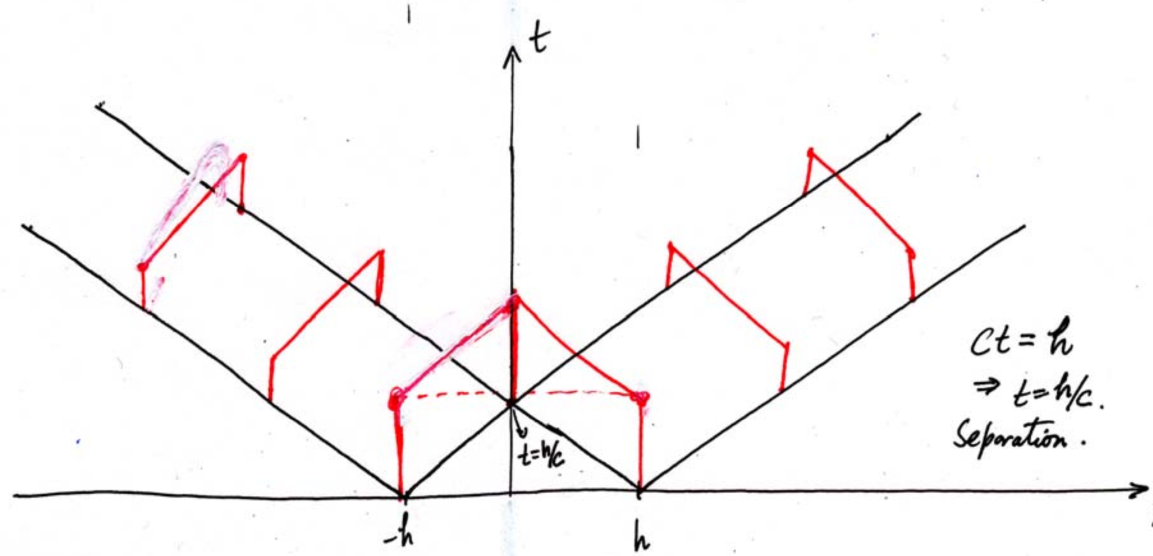
In Region IV: $x-ct < -h$ and $x+ct > h$.

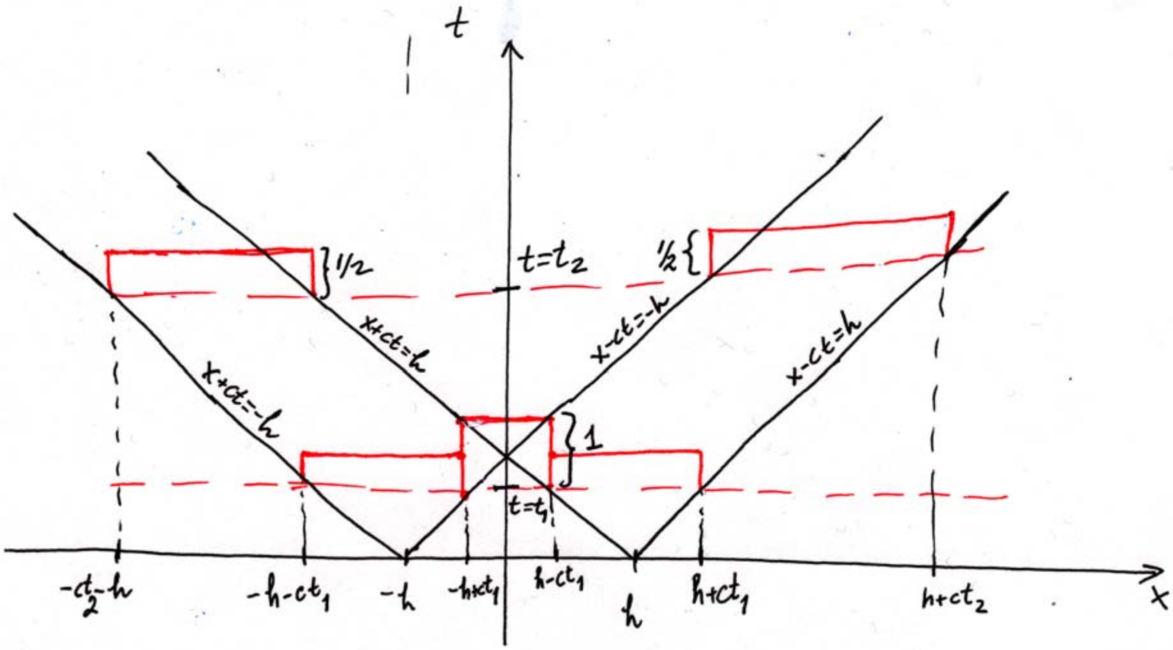
$$u(x,t) = \frac{1}{2}\left[0 + 0 \right] = 0.$$

In Region V: $x-ct < -h$ and $x+ct < -h$. $\Rightarrow u(x,t) = \frac{1}{2}[0+0] = 0$

In Region VI: $x-ct > h$ and $x+ct > h \Rightarrow u(x,t) = \frac{1}{2}[0+0] = 0.$


t
IV
u=0
III
x+ct=h
x-ct=-h
II
V
x+ct=-h
x-ct=h
VI
u=0
u=1/2
t=h/c
u=1/2
u=0
I
u=1
x
-h
h


t
ct=h
⇒ t=h/c.
Separation.
t=h/c
x
-h
h


t
t=t2
1/2
1/2
x+ct=h
x-ct=-h
x+ct=-h
x-ct=h
1
t=t1
-ct2-h
-h-ct1
-h
-h+ct1
h-ct1
h
h+ct1
h+ct2
x

Example.- Solve the IVP.

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \quad t > 0.$$

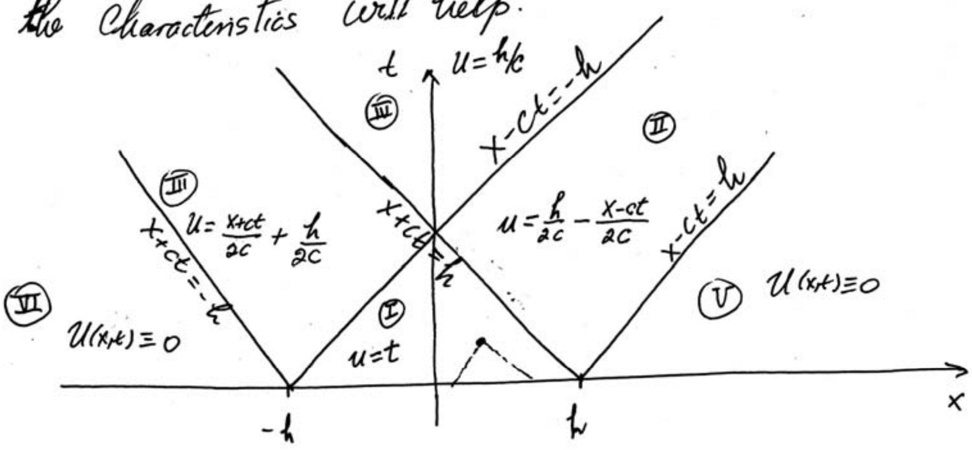
$$u(x,0) = f(x) \equiv 0, \qquad u_t(x,0) = \begin{cases} 1, & |x| < h \\ 0, & |x| > h. \end{cases} \equiv g(x)$$

D'Alambert formula

$$u(x,t) = \frac{1}{2}\left[\cancel{f(x+ct)}^{\,0} + \cancel{f(x-ct)}^{\,0} \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\, d\bar{x}$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\, d\bar{x}$$

To obtain $u(x,t)$ a graphic of the xt-plane and the characteristics will help.



In Region (I): $-h < x-ct$ and $x+ct < h$

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\, d\bar{x} = \frac{1}{2c}\int_{x-ct}^{x+ct} 1\, d\bar{x} = \frac{1}{2c}[x+ct-x+ct] = \frac{2ct}{2c} = t.$$

$$\Rightarrow \boxed{u(x,t) = t}$$

In Region (II): $-h < x-ct < h$ and $x+ct > h$.

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_{x-ct}^{h} \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_{h}^{x+ct} 0\,d\bar{x}$$

$$\therefore \quad \boxed{u(x,t) = \frac{h-x+ct}{2c} = -\frac{(x-ct)}{2c} + \frac{h}{2c}}$$

In Region (IV): $x-ct < -h$ and $x+ct > h$.

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_{x-ct}^{-h} \overset{\equiv 0}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_{-h}^{h} \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_{h}^{x+ct} \overset{\equiv 0}{g(\bar{x})}\,d\bar{x}$$

$$\therefore \quad \boxed{u(x,t) = \frac{h-(-h)}{2c} = \frac{h}{c}}$$

In Region (III): $x-ct < -h$ and $-h < x+ct < h$.

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_{x-ct}^{-h} \overset{\equiv 0}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_{-h}^{x+ct} \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} +$$

$$\therefore \quad \boxed{u(x,t) = \frac{1}{2c}\left[\frac{x+ct}{2c} + \frac{h}{2c}\right] = \frac{(x+ct)}{2c} + \frac{h}{2c}}$$

In Region (V): $x-ct > h$ and $x+ct > h$

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \overset{=0}{g(\bar{x})}\,d\bar{x} = 0. \quad \Rightarrow \quad \boxed{u(x,t) \equiv 0.}$$

In Region VI: $x-ct < -h$ and $x+ct < -h$

$$\Rightarrow \quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \overset{=0}{g(\bar{x})}\,d\bar{x} = 0 \quad \Rightarrow \quad \boxed{u(x,t) \equiv 0}$$

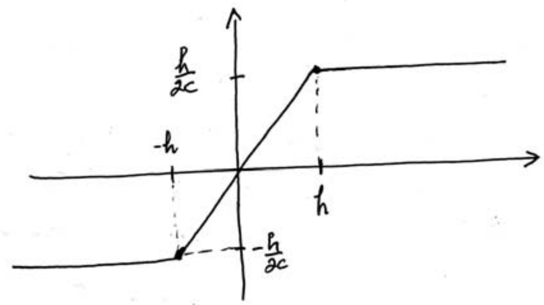
Graphing the time evolution of the wave equation for our last example.

1) Graph of

$$\frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = \frac{1}{2c}\begin{cases} \text{If } x<-h, & \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = -\frac{1}{2c}\int_x^{-h} \overset{\equiv 0}{g(\bar{x})}\,d\bar{x} - \frac{1}{2c}\int_{-h}^0 \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} = \frac{-h}{2c} \\ \text{If } -h<x<, & \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = -\frac{1}{2c}\int_x^0 \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} = \underline{\underline{\frac{1}{2c}x}} \\ \text{If } 0<x<+h, & \frac{1}{2c}\int_0^x \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} = \frac{1}{2c}x. \\ \text{If } x>h, & \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_0^h \overset{\equiv 1}{g(\bar{x})}\,d\bar{x} + \frac{1}{2c}\int_h^x \overset{\equiv 0}{g(\bar{x})}\,d\bar{x} = \frac{h}{2c} \end{cases}$$

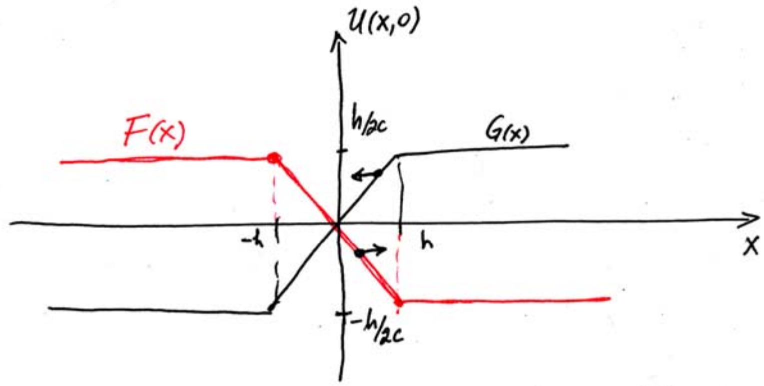
Summarizing:

$$\frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x} = \begin{cases} -\frac{h}{2c}, & x<-h \\ \frac{x}{2c}, & -h<x<h. \\ \frac{h}{2c}, & x>h. \end{cases}$$



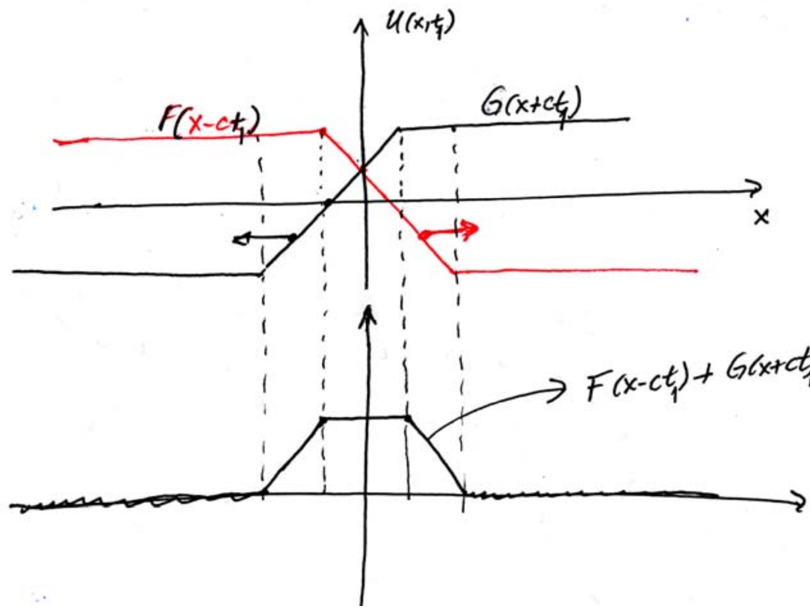
2) Form $F(x)$ and $G(x)$.

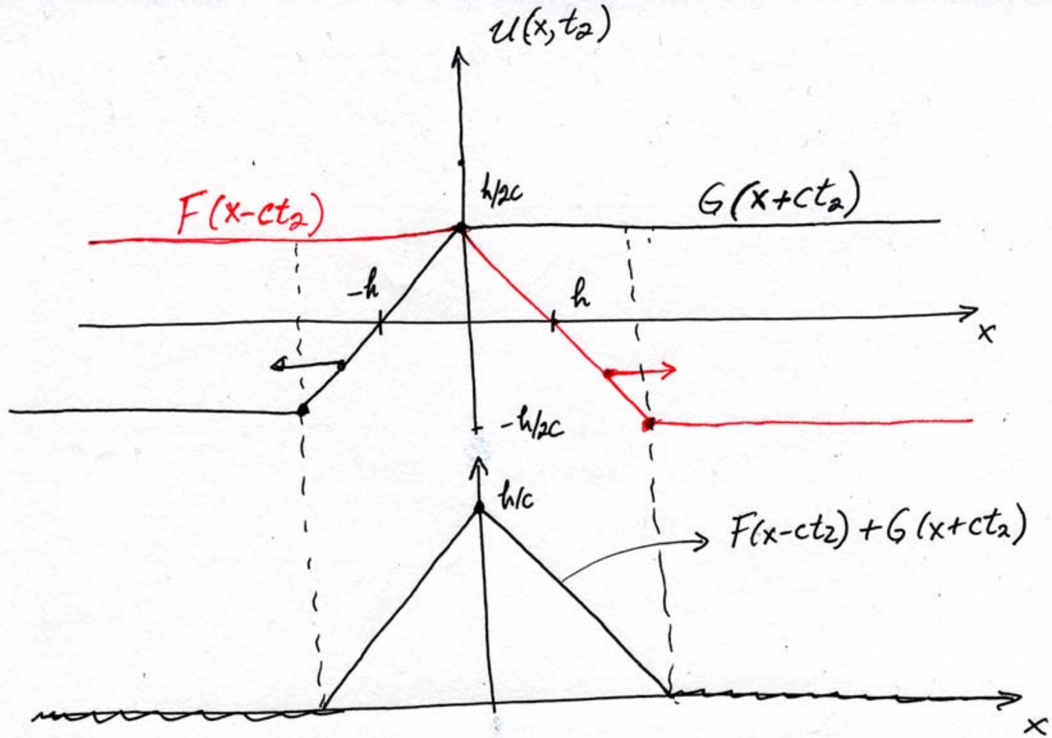
$$F(x) = -\frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x}, \qquad G(x) = \frac{1}{2c}\int_0^x g(\bar{x})\,d\bar{x}$$



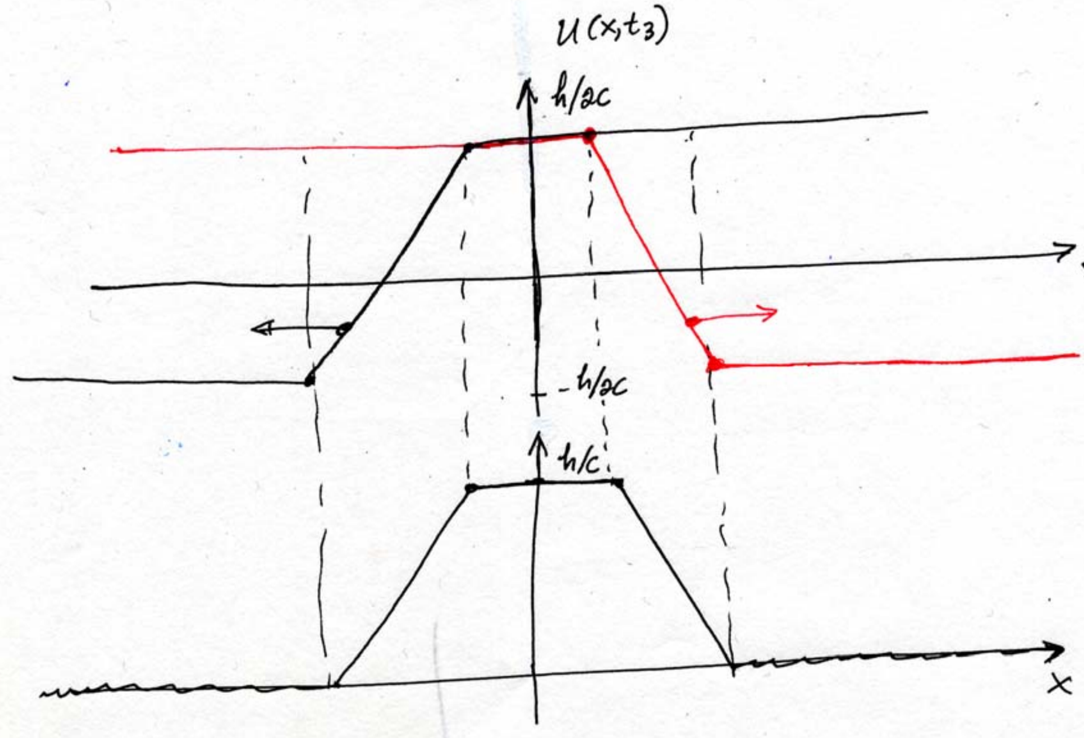
$$u(x,0) = F(x) + G(x) \equiv 0. \quad \text{for all } x$$

3), 4) Translate $F(x)$ to the right $\rightarrow F(x-ct)$
Translate $G(x)$ to the left $\rightarrow G(x+ct)$. $\Rightarrow$ Add the two functions to obtain $u(x,t) = F(x-ct) + G(x+ct)$





$u(x,t_2)$
$h/2c$
$F(x-ct_2)$
$G(x+ct_2)$
$-h$
h
x
$-h/2c$
h/c
$F(x-ct_2)+G(x+ct_2)$
x



$u(x,t_3)$
$h/2c$
x
$-h/2c$
h/c
x

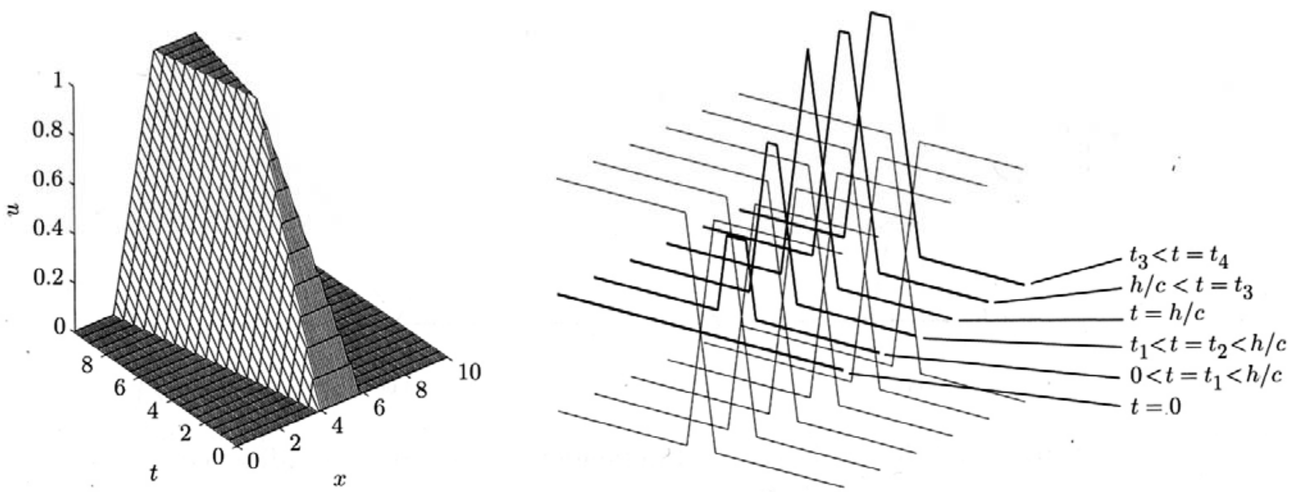


Figure 12.3.4 Time evolution for a struck string.