MATH 411 Chapter 5

0.1 Definition 1: Lipschitz Condition

A function $f(t, y)$ is said to satisfy a Lipschitz condition in the variable y on a set $D \subseteq \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

whenever (t, y_1), $(t, y_2) \in D$. The constant L is called a Lipschitz constant for f.

0.2 Definition 2: Convex Set

A set $D \subseteq \mathbb{R}^2$ is said to be convex if for all (t, y_1) and $(t, y_2) \in D$, the point $((1 - \lambda)t_1 + \lambda t_2, (1 - \lambda)y_1 + \lambda y_2) \in D$, for every $\lambda \in [0, 1]$.

0.3 Theorem 1: Sufficient condition

Assume $f(t, y)$ is defined in a convex set $D \subseteq \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L,$$

for all $(t, y) \in D$, then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L.

0.4 Theorem 2: Existence and Uniqueness of Solutions

- $D = \{(t, y) | t \in [a, b], y \in (-\infty, \infty)\}$.
- $f(t, y)$ continuous on D and satisfies a Lipschitz condition on D in the variable y, then the initial value problem (IVP)

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha$$

has a unique solution $y(t)$ for $t \in [a, b]$

0.5 Theorem 2.2: Existence and Uniqueness of Solutions (more general)

- $D = \{(t, y) | t \in [a, b], y \in (\alpha - d, \alpha + d)\}$.
- $f(t, y)$ satisfies a Lipschitz condition on D in the variable y, then the initial value problem

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha$$

has a unique solution $y(t)$ for $t \in [a, T]$, where $T = \min(b, a + d/M)$ with $M = \max_{(t,y)\in D} |f(t, y)|$

Remark: M is the maximum modulus of the slope that $y(t)$ can reach in the time interval, so that up to time T we know that $y(t)$ remains in the domain D where the Lipschitz condition is satisfied.

0.6 Definition 3: Well-posed problem

The IVP

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha$$

is said to be well-posed problem if:

- A unique solution $y(t)$ for the IVP exists for $t \in [a, b]$
- For any ϵ, there is a positive constant $k(\epsilon)$ and a unique solution $z(t)$ for the perturbed problem

  $$z'(t) = f(t, z) + \delta(t), \quad t \in [a, b], \quad z(a) = \alpha + \epsilon_0$$

  if $\delta(t)$ is continuous and $\delta(t) < \epsilon$ on $[a, b]$, and $|\epsilon_0| < \epsilon$. Also, $z(t)$ satisfies

  $$|z(t) - y(t)| < k(\epsilon)\epsilon$$

0.7 Theorem 3: Condition for well-posed problem

- $D = \{(t, y) | t \in [a, b], y \in (-\infty, \infty)\}$.
- $f(t, y)$ continuous on D and satisfies a Lipschitz condition on D in the variable y, then the initial value problem (IVP)

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha$$

is well-posed.

0.8 Euler Method

Consider the IVP

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \tag{1}$$

$t_0 = a, \quad t_N = b$ and $t_{i+1} = t_i + h, \quad i = 0, 1, \dots N - 1.$
Euler Method:
$w_0 = \alpha,$
$w_{i+1} = w_i + hf(t_i, w_i), \quad i = 0, 1 \dots N - 1$

0.8.1 Theorem 5.9(book): Bound for Global Error

- f continuous and satisfies Lipschitz condition on $D = \{(t, y)|t \in [a, b], y \in (-\infty, \infty)\}$.
- There is M such that $|y''(t)| \leq M$, for all $t \in [a, b]$
- $y(t)$ unique solution for (1).
- $w_0, w_1, \dots w_N$ the approximations of Euler Method.

Then,

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right] \tag{2}$$

1 Section 5.10: Stability and Convergence

Consider the IVP

$$y'(t) = f(t, y), \quad t \in [a, b], \quad y(a) = \alpha \tag{3}$$

and the one-step finite difference method (FDM)

$$w_o = \alpha, \qquad w_{i+1} = w_i + h\phi(t_i, w_i, h), \quad i = 0 \dots, N - 1 \tag{4}$$

with local truncation error (LTE) at the ith step $\tau_i(h)$.

1.1 Definition: Consistency

The FDM (4) is consistent with the IVP (3) if

$$\lim_{h\to 0} ||\tau(h)||_\infty = \lim_{h\to 0} \max_{1\leq i\leq N} |\tau_i(h)| \to 0,$$

where $\tau(h) = \begin{pmatrix} \tau_1(h) \\ \tau_2(h) \\ . \\ \tau_N(h) \end{pmatrix}$

1.2 Definition: Convergence

The FDM (4) is convergent with respect to the IVP (3) if

$$\lim_{h\to 0} \max_{1\le i\le N} |w_i - y(t_i)| \to 0$$

1.3 Definition: Stability

The FDM (4) is stable if given two solutions $\{u_i\}_0^N$ and $\{v_i\}_0^N$ such that their initial values satisfy $|u_0 - v_0| < \epsilon$, then there exists $K > 0$ such that

$$|u_i - v_i| < K\epsilon, \qquad 1 \le i \le N$$

It means that small changes in the initial conditions produce correspondingly small changes in the subsequent approximations. Compare with well-posedness of an IVP.

1.4 Theorem 5.20: Stability-Convergence-Consistency

If there is $h_0 > 0$ such that $\phi(t, w, h)$ of FDM (4) is continuous and satisfies a Lipschitz condition in w with constant L on the set

$$D = \{(t, w, h) \mid t \in [a, b], w \in (-\infty, \infty), 0 \le h \le h_0\}$$

Then,

1. The method is stable.
2. The FDM is convergent if and only if it is consistent. This is equivalent to

$$\phi(t, y, 0) = f(t, y), \qquad a \le t \le b$$

3. If a function $\eta(h)$ exists such that $|\tau_i(h)| \le \eta(h)$ when $0 \le h \le h_0$, then

$$|y(t_i) - w_i| \le \frac{\eta(h)}{L} e^{L(t_i - a)}$$

1.5 Consistency and Convergence of Multistep Methods

Consider a general m-step multistep method

$$\begin{aligned} &w_0 = \alpha, \quad w_1 = \alpha_1, \ldots \quad w_{m-1} = \alpha_{m-1}, \\ &w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \ldots a_0 w_{i-(m-1)} + hF(t, h, w_{i+1}, w_i \ldots w_{i-(m-1)}), \end{aligned} \tag{5}$$

for $i = m-1, m, \ldots N-1$.

1.6 Definition: Local Truncation error

If $y(t_i) = y_i$, for $i = 0, \dots N$ are the values of the exact solution evaluated at the grid points, then the local truncation error is defined as

$$\tau_{i+1}(h) = \frac{y_{i+1} - a_{m-1}y_i - \cdots - a_0 y_{i-(m-1)}}{h} - F(t, h, y_{i+1}, y_i \dots y_{i-(m-1)}), \tag{6}$$

for $i = m-1, m, \dots N-1$.

1.7 Definition: Lipschitz Condition for Multistep Methods

The function F in the definition of multistep methods (5) satisfies a Lipschitz condition with respect to the sequence $\{w_j\}_0^N$, if there is $L > 0$ such that for every pair of sequences $\{v_j\}_0^N$, $\{\tilde{v}_j\}_0^N$

$$|F(t_i, h, v_{i+1}, \dots, v_{i-(m-1)}) - F(t_i, h, \tilde{v}_{i+1}, \dots, \tilde{v}_{i-(m-1)})| \le L \sum_{j=0}^{m} |v_{i+1-j} - \tilde{v}_{i+1-j}| \tag{7}$$

for $i = m-1, \dots N-1$.

1.8 Definition: Consistency for Multistep Methods

The multistep method (5) is consistent if

$$\lim_{h \to 0} |\tau_i(h)| = 0, \quad i = m, \dots N$$

and

$$\lim_{h \to 0} |\alpha_i - y_i| = 0, \quad i = 1, \dots m-1.$$

This last condition may be interpreted as requiring convergence of the “starting method”.

1.9 Theorem 5.21: Local Truncation Error, Consistency and Convergence for Adams Multistep Methods

Consider the IVP

$$y' = f(t, y), \qquad a \leq t \leq b, \qquad y(a) = \alpha$$

which is approximated by an m-step explicit predictor-corrector Adam's method. If

1. the predictor part has local truncation error $\tau_{i+1}(h)$ and the corrector part has LTE $\tilde{\tau}_{i+1}(h)$,
2. f and $\frac{\partial f}{\partial y}$ are continuous on $D = \{(t, y) | t \in [a, b], y \in (-\infty, \infty)\}$, and $\frac{\partial f}{\partial y}$ is bounded on D. Therefore, f satisfies a Lipschitz condition on D.

Then,

1. the local truncation error of the predictor-corrector is given by

   $$\sigma_{i+1}(h) = \tilde{\tau}_{i+1}(h) + \tau_{i+1}(h)\, \tilde{b}_{m-1} \frac{\partial f}{\partial y}(t_{i+1}, \theta_{i+1}), \tag{8}$$

   where θ_{i+1} is between 0 and $h\tau_{i+1}(h)$.
2. There are constants k_1 and k_2 such that

   $$|w_i - y(t_i)| \leq \left[\max_{0 \leq j \leq m-1} |w_j - y_j| + k_1 \sigma(h) \right] e^{k_2(t_i - a)}, \tag{9}$$

   where $\sigma(h) = \max_{m \leq j \leq N} |\sigma_j(h)|$

Discuss this theorem.

1.10 Definition: Characteristic Polynomial

The characteristic polynomial corresponding to (5) is defined as

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0$$

1.11 Definition: Root Condition

If $\lambda_1, \dots \lambda_m$ are the roots of the characteristic polynomial

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_1\lambda - a_0$$

corresponding to the multistep method (5),

i) $|\lambda_i \leq 1 \qquad i = 1, \dots m$ and

ii) all roots with absolute value 1 are simple roots.

Then, the difference method is said to satisfy a **root condition**.
Discuss this concept.

1.12 Definition: Stability

1. Multistep methods that satisfy the root condition and have $\lambda = 1$ as the only root of the characteristic equation of magnitude 1 are called strongly stable.
2. Multistep methods that satisfy the root condition and have more than one distinct root with magnitude 1 are called weakly stable.
3. Multistep methods that do not satisfy the root condition are called unstable.

1.13 Theorem 5.24: Stability, Consistency, and Convergence of General Multistep Methods

If the m-step multistep method

$$\begin{aligned} &w_0 = \alpha, \quad w_1 = \alpha_1, \dots \quad w_{m-1} = \alpha_{m-1}, \\ &w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \dots a_0 w_{i-(m-1)} + hF(t, h, w_{i+1}, w_i \dots w_{i-(m-1)}), \end{aligned} \tag{10}$$

for $i = m-1, m, \dots N-1$, is consistent then,
it is stable if and only if it is convergent.
Discuss it.