11.3 FINITE DIFFERENCES METHODS FOR LINEAR PROBLEMS.

In Sections 11.1–11.2

BVPs $\longrightarrow$ IVPs.

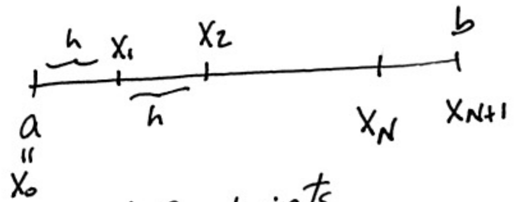
Now, we will solve the BVP directly. Each derivative present in

$$\text{BVP} \begin{cases} y''(x) = p(x)y' + q(x)y + r(x), & a<x<b \quad (1) \\ y(a) = \alpha, \quad y(b) = \beta \ \text{ or } \ y'(a)=\alpha, \ y'(b)=\beta & (2) \end{cases}$$

or comb. of both.

will be approx. using finite differences.

(I) First a uniform partition of the interval $[a,b]$ is performed



$N+2$ - points
$N+1$ - subintervals

x_i

$h = \dfrac{b-a}{N+1}$ ← $N+1$ TOTAL # of points minus 1.

At each interior point x_i, $i = 1, .., N$

we want to approx. the derivatives involved in (1).

For example,

$$y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + \underbrace{\frac{-h^2}{12} y''''(\eta_i)}_{= \mathcal{O}(h^2)}$$

y'''' bounded. $h \to 0$

$\eta_i \in (x_{i-1}, x_{i+1})$.

Therefore,

$$y''(x_i) = \underbrace{\frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2}}_{\substack{\text{Centered finite difference} \\ \text{approx. of } y''(x_i).}} + \tilde{\tau}_i(h) \qquad (3)$$

where $\tau_i(h) = -\frac{h^2}{12} y''(\eta_i)$, $\eta_i \in (x_{i-1}, x_{i+1})$

is called the truncation error in the approx. of $y''(x_i)$ by (3)

Similarly,

$$y'(x_i) = \underbrace{\frac{y(x_{i+1}) - y(x_{i-1})}{2h}}_{\substack{\text{Centered finite} \\ \text{difference approx. of } y'(x_i).}} \underbrace{- \frac{h^2}{6} y'''(\hat{\eta}_i)}_{\tau_i(h)} \qquad (4)$$

$\hat{\eta}_i \in (x_{i-1}, x_{i+1})$

Proof of (3)

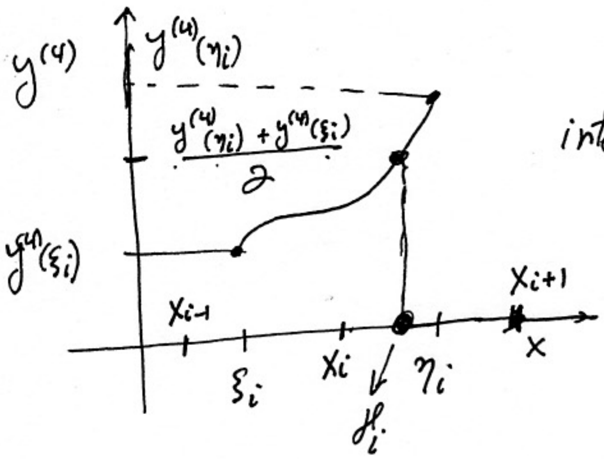
$$y(x_{i+1}) = y(x_i) + h\, y'(x_i) + \frac{h^2}{2} y''(x_i) + \frac{h^3}{3!} y'''(x_i) + \frac{h^4}{4!} y^{(4)}(\gamma_i) \qquad (4.2)$$

$\gamma_i \in (x_i, x_{i+1})$

$$y(x_{i-1}) = y(x_i) - h\, y'(x_i) + \frac{h^2}{2} y''(x_i) - \frac{h^3}{3!} y'''(x_i) + \frac{h^4}{4!} y^{(4)}(\xi_i) \qquad (4.3)$$

$\xi_i \in (x_{i-1}, x_i)$

$$\Rightarrow y(x_{i+1}) + y(x_{i-1}) = 2y(x_i) + h^2 y''(x_i) + \frac{h^4}{24}\left[y^{(4)}(\gamma_i) + y^{(4)}(\xi_i) \right]$$



intermediate value theorem (IVT)

$$+\frac{h^2}{12}\left[\frac{y^{(4)}(\gamma_i)+y^{(4)}(\xi_i)}{2}\right] \xrightarrow{IVT}$$

Therefore,
$$\boxed{\frac{y(x_{i+1})-2y(x_i)+y(x_{i-1})}{h^2} = y''(x_i)+\frac{h^2}{12}y^{(4)}(\eta_i)} \quad (4)$$
$\eta_i \in (x_{i-1}, x_{i+1})$

Proof of (4): From (4.2)-(4.3)

$$y(x_{i+1}) - y(x_{i-1}) = 2h\, y'(x_i) + \frac{h^3}{3!}\left(y'''(\hat{\gamma}_i)+y'''(\hat{\xi}_i)\right)$$

$\hat{\gamma}_i \in (x_i, x_{i+1})$
$\hat{\xi}_i \in (x_{i-1}, x_i)$

$$\frac{h^3}{3}\left(\frac{y^{(3)}(\hat{\gamma}_i)+y^{(3)}(\hat{\xi}_i)}{2}\right) \overset{IVT}{=\!=} \frac{h^3}{3}y^{(3)}(\hat{\eta}_i)$$

$\hat{\eta}_i \in (x_{i-1}, x_{i+1})$

$$\Rightarrow \quad \boxed{y'(x_i) = \frac{y(x_{i+1})-y(x_{i-1})}{2h}\underbrace{-\frac{h^2}{6}y^{(3)}(\hat{\eta}_i)}_{=O(h^2)}} \quad (5)$$

Subst. of (4)-(5) into (1) for $x = x_i$

$$\frac{y(x_{i+1})-2y(x_i)+y(x_{i+1})}{h^2} = p(x_i)\left[\frac{y(x_{i+1})-y(x_{i-1})}{2h}\right] +$$
$$+\, q(x_i)y(x_i) + r(x_i) - \frac{h^2}{12}\left[2p(x_i)\, y^{(3)}(\hat{\eta}_i) - y^{(4)}(\eta_i)\right]$$

Assuming $y \in C^4[a,b] \Rightarrow y^{(4)}(\eta_i)$ and $y^{(3)}(\hat{\eta}_i)$ are bounded as $h \to 0$.

$$\tau_i(h) = -\frac{h^2}{12}\left[2p(x_i)\, y^{(3)}(\hat{\eta}_i) - y^{(4)}(\eta_i)\right] \xrightarrow[h\to 0]{} 0$$

Local Trunc. error (LTE)

Dropping the L.T.E. and approx. $w_i \simeq y(x_i)$

The <u>Finite-Difference method</u> (FDM) is obtained

$$\boxed{-\left(\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}\right) + p(x_i)\left(\frac{w_{i+1} - w_{i-1}}{2h}\right) + q(x_i)\, w_i = -r(x_i) \qquad i = 1, 2, \ldots, N.} \quad (6)$$

It can also be written as (multiplying by h^2 and reordering)

$$\boxed{-\left(1 + \frac{h}{2}p(x_i)\right)w_{i-1} + \left(2 + h^2 q(x_i)\right)w_i + -\left(1 - \frac{h}{2}p(x_i)\right)w_{i+1} = -h^2 r(x_i)} \quad (7)$$

$i=1$ $\quad -\left(1+\frac{h}{2}p_1\right)\overset{=\alpha}{w_0} + (2+h^2 q_1)w_1 - \left(1-\frac{h}{2}p_1\right)w_2 = -h^2 r_1$

$i=2$ $\quad -\left(1+\frac{h}{2}p_2\right)w_1 + (2+h^2 q_2)w_2 - \left(1-\frac{h}{2}p_2\right)w_3 = -h^2 r_2$

$\vdots$

$i=N$ $\quad -\left(1+\frac{h}{2}p_N\right)w_{N-1} + (2+h^2 q_N)w_N - \left(1-\frac{h}{2}p_N\right)\overset{=\beta}{w_{N+1}} = -h^2 r_N$

It can be written as $\boxed{A\vec{w} = \vec{b}}$ $\quad$ (8)

where

$$A = \begin{bmatrix} 2+h^2 q_1 & -1+\frac{h}{2}p_1 & 0 & 0 & \cdots & 0 \\ -1-\frac{h}{2}p_2 & 2+h^2 q_2 & -1+\frac{h}{2}p_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -1-\frac{h}{2}p_N & 2+h^2 q_N \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -h^2 r_1 + \left(1+\frac{h}{2}p_1\right) w_0^{=\alpha} \\ -h^2 r_2 \\ \vdots \\ -h^2 r_{N-1} \\ -h^2 r_N + \left(1-\frac{h}{2}p_N\right) w_{N+1}^{=\beta} \end{bmatrix}$$

<u>Thm (Unique Soln. Tridiag. Syst. (8))</u>

1) $p(x), q(x), r(x)$ conts on $[a,b]$.
2) $q(x) \geq 0$ on $[a,b]$.
3) $h < 2/L$, $\quad L \equiv \max_{a \leq x \leq b} |p(x)|$.

$\Rightarrow$ Syst. (8) has a unique soln.