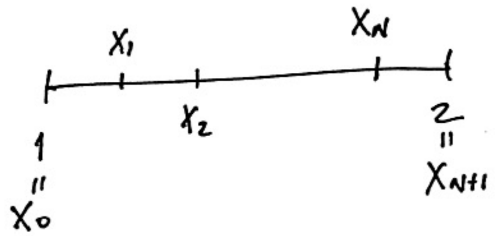
11.4 FINITE DIFFERENCE METHODS FOR Nonlinear PROBLEMS.

Consider the nonlinear BVP:

Probl. 1) in section 11.4.

$$\begin{cases} y''(x) + (y'(x))^2 + y(x) = \ln x, & 1 < x < 2 \quad (1) \\ y(1) = 0, \quad y(2) = \ln(2). & \quad (2) \end{cases}$$

Grid points:



$$h = \frac{2-1}{N+1} = \frac{1}{N+1}$$

$$x_i = 1 + ih$$

Using centered difference approx for (1)

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + \left[\frac{w_{i+1} - w_{i-1}}{2h}\right]^2 + w_i = \ln(x_i)$$

From here the following nonlinear system is obtained,

$i=1$:

$$f_1(w_1, w_2, \ldots, w_N) = \frac{w_2 - 2w_1 + w_0}{h^2} + \left[\frac{w_2 - w_0}{2h}\right]^2 + w_1 - \ln(x_1) = 0$$

$i=2$:

$$f_2(w_1, w_2, \ldots, w_N) = \frac{w_3 - 2w_2 + w_1}{h^2} + \left[\frac{w_3 - w_1}{2h}\right]^2 + w_2 - \ln(x_2) = 0 \qquad (3)$$

$\vdots$

$$i: f_i(w_1, w_2, \ldots, w_N) = \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} + \left[\frac{w_{i+1} - w_{i-1}}{2h}\right]^2 + w_i - \ln(x_i) = 0$$

$\vdots$

$$i = N: f_N(w_1, w_2, \ldots, w_N) = \frac{w_{N+1} - 2w_N + w_{N-1}}{h^2} + \left[\frac{w_{N+1} - w_{N-1}}{2h}\right]^2 + w_N - \ln(x_N) = 0$$

The BVP reduces to find the roots for the vector equation

$$\boxed{F(\vec{\omega}) = \left(f_1(\vec{\omega}), \dots f_N(\vec{\omega})\right) = \vec{0}} \qquad (4)$$

where $\vec{\omega} \equiv (\omega_1, \omega_2, .., \omega_N)$.

Using Newton method, we transform (4) into a fixed point problem for

$$\boxed{G(\vec{w}) = \vec{w}}$$

where

$$\boxed{G(\vec{\omega}) \equiv \vec{\omega} - J^{-1}(\vec{\omega})\vec{F}(\vec{\omega})} \qquad (5)$$

and

$$J(\vec{\omega}) \equiv \begin{bmatrix} \frac{\partial f_1}{\partial \omega_1} & \frac{\partial f_1}{\partial \omega_2} & \cdots & \frac{\partial f_N}{\partial \omega_N} \\ \frac{\partial f_2}{\partial \omega_1} & \frac{\partial f_2}{\partial \omega_2} & \cdots & \frac{\partial f_N}{\partial \omega_N} \\ \vdots & & & \\ \frac{\partial f_N}{\partial \omega_1} & \frac{\partial f_N}{\partial \omega_2} & \cdots & \frac{\partial f_N}{\partial \omega_N} \end{bmatrix}$$

Jacobian matrix of vector function $F(\vec{\omega})$.

Equ. (5) is used for a functional iteration,

$$G(\vec{\omega}^{(k)}) = \vec{\omega}^{k-1} - J^{-1}(\vec{\omega}^{k-1})\vec{F}(\vec{\omega}^{k-1})$$

Knowing $\vec{\omega}^{k-1}$, the new approx. $\vec{\omega}^{k}$ is obtained in two steps:

$$\boxed{\begin{array}{ll} \text{Step 1:} & J(\vec{\omega}^{k-1})\,\vec{y} = \vec{F}(\vec{\omega}^{k-1}) \\ \text{Step 2:} & \vec{x}^{(k)} = \vec{x}^{(k-1)} + \vec{y} \end{array}}$$

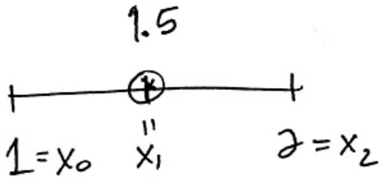
So a key part in the computation is the derivation of the Jacobian, $J(\vec{w})$ of $\vec{F}(\vec{w})$.

In our particular example,

$$
J(\vec{w}) = \begin{bmatrix}
\left(-\frac{2}{h^2}+1\right) & \left(\frac{1}{h^2}+\frac{w_2-w_0}{h}\right) & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\left(\frac{1}{h^2}-\frac{w_3-w_1}{h}\right) & \left(-\frac{2}{h^2}+1\right) & \left(\frac{1}{h^2}+\frac{w_3-w_1}{h}\right) & 0 & \cdots & & & 0 \\
0 & \ddots & & & & & & \\
\vdots & & & & \left(\frac{1}{h^2}-\frac{w_N-w_{N-2}}{h}\right) & \left(-\frac{2}{h^2}+1\right) & \left(\frac{1}{h^2}+\frac{w_N-w_{N-2}}{h}\right) & \\
0 & \cdots & & & & & \left(\frac{1}{h^2}-\frac{w_{N+1}-w_{N-1}}{h}\right) & \left(-\frac{2}{h^2}+1\right)
\end{bmatrix}
$$

$$
F(\vec{w}) = \begin{bmatrix}
\frac{w_2-2w_1-w_0}{h^2}+\left(\frac{w_2-w_0}{2h}\right)^2+w_1-\ln(x_1) \\
\frac{w_3-2w_2+w_1}{h^2}+\left(\frac{w_3-w_1}{2h}\right)^2+w_2-\ln(x_2) \\
\vdots \\
\frac{w_{N+1}-2w_N+w_{N-1}}{h^2}+\left(\frac{w_{N+1}-w_{N-1}}{2h}\right)^2+w_N-\ln(x_N)
\end{bmatrix}
$$

If $h=\frac{1}{2}$



$x_1 = 1+\frac{1}{2} = \frac{3}{2} = 1.5$ is the only point where we approximate the soln. $y(x)$ for (1)-(2). , $N=1$

$$f_1(w_0,w_1,w_2) = \frac{w_2-2w_1+w_0}{h^2=\frac{1}{4}} + \left(\frac{w_2-w_0}{1}\right)^2 + w_1 - \ln(1.5) = 0.$$

Now, $w_0=0$, $w_2=\ln(2)$

$$\frac{\ln(2)-2w_1}{1/4} + \ln^2(2) + w_1 - \ln(1.5) = 0$$

$$\ln(2)-2w_1+\frac{1}{4}\left[\ln^2(2)+w_1-\ln(1.5)\right]=0$$

or $$\frac{1}{4}w_1 - 2w_1 = -\ln(2) - \frac{1}{4}\left(\ln^2(2)-\ln(1.5)\right)$$

or $$w_1 = \frac{-\ln(2)-\frac{1}{4}\left(\ln^2(2)-\ln(1.5)\right)}{-7/4} \simeq 0.407.$$

In general, consider the BVP

$$\begin{cases} y'' = f(x,y,y'), & a<x<b \qquad (6) \\ y(a)=\alpha, \quad y(b)=\beta & \qquad (7) \end{cases}$$

Theorem 11.1 (Book)

If

1) f is cont's on $D = \{(x,y,y') \mid a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\}$
2) f_y and $f_{y'}$ are also cont's on D.
3) $f_y(x,y,y') > 0$, for all $(x,y,y') \in D$.
4) There exists such that:

$$|f_{y'}(x,y,y')| \le M, \quad \text{for all } (x,y,y') \in D$$

Then, the BVP (6)-(7) has a unique soln.

In our example,

$$D = \left\{ (x,y,y') \;\middle|\; \begin{array}{l} 1<x<2 \\ -\infty<y<\infty \\ -\infty<y'<\infty \end{array} \right\}$$

$$f(x,y,y') = -(y')^2 - y + \ln(x)$$

$$f_y(x,y,y') = -1 < 0$$

$$f_{y'}(x,y,y') = -2y'$$

→ There is no assurance of a unique soln.