Math 411-01

Final Project

Name: ______________________________

Due before 6 pm, Friday April 17, 2009

Answer all questions and show all your work carefully. This is an individual test. You should not discuss the questions and your answers with any one. *BYU students should seek to be totally honest in their dealings with others. They should complete their own work and be evaluated based upon that work. (Honor Code)*

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Problem No.	**Points**
a)	
b)	
c)	
d)	
e)	
f)	
Total	

Consider the initial boundary value problem (IBVP) of the start-up of pressure-driven laminar flow in an infinite channel of square cross section determined by solving the dimensionless equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 1, \qquad t > 0, \tag{1}$$

subject to the initial condition: $\quad u(x, y, 0) = 0$

and the no-slip BC's: $u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0$.
The exact solution can be obtained by eigenfunctions expansions and is given by

$$u(x, y, t) = \frac{16}{\pi^4} \sum_{k,l=0}^{\infty} \frac{\sin(\pi(1 + 2k)x) \sin(\pi(1 + 2l)y)}{(1 + 2l)(1 + 2k)\left[(1 + 2k)^2 + (1 + 2l)^2\right]} \left(1 - e^{-\pi^2\left((1+2k)^2+(1+2l)^2\right)t}\right) \tag{2}$$

This project consists of obtaining numerical approximations to this exact solution using the predictor-corrector ADI method.

a) Derive the finite-difference equations to be used at the predictor step. For this non-homogeneous equation, try to imitate the derivation performed in class (see notes posted in our course web-page) for the homogeneous equation. Similarly, derive the finite-difference equations to be used at the corrector step.

b) Write the predictor and corrector equations corresponding to the application of the ADI method to the above IBVP in matrix form. First, write a sequence of scalar equations and then represent the predictor and corrector equations as linear systems, respectively. Clearly, define all matrices and all vectors involved in the equations.

c) As shown in class, you should notice that the above linear systems are tridiagonal. Write your own tridiagonal algorithm, TRIDIAG( ), based on Crout factorization (Burden-Faires pp. 407-409). You should call your TRIDIAG subroutine at every needd step from your ADI solver. Write a separate program for TRIDIAG( ). Your TRIDIAG( ) subroutine should be handed in with the rest of your ADI code, graphs, and your results.

d) Write a computer program for the ADI application to the above IBVP. Use a uniform partition of 25 interior points in the x-direction, 30 interior points in the y-direction, and a time step $\Delta t = 0.01$. "Visualize" (using contour plots, or 3-D surface plots) the velocity field at the following times: $t_n = 0.05, 0.1, 0.2, 0.5, 1$, and 2. For comparison, draw the same set of plots for the exact solution at the same times.

e) Compute the maximum velocity $u(x, y, t_n)$ at each of the times t_n of part (c). Identify the point(s) where this maximum velocity is obtained in each case. Verify that your numerical results are "good approximations" to the exact solution by obtaining $u(x, y, t_n)$ directly from (2) at the same points. Use 30 points for each index (k and l) to truncate the infinite series defining $u(x, y, t)$.

f) Compare, in the L_2 norm (Euclidean norm) and the infinity norm, each one of the velocity fields obtained at time $t_n = 0.05, 0.1, \cdots 2$ with the velocity fields at the previous time $t_n - \Delta t = t_{n-1}$, respectively. Then, compare the velocity field at time $t_n = 2$ with the the velocity field at time $t_n = 4$ and $t_n = 5$, respectively. You should notice the behavior of the solution when t grows. In your own words, describe this behavior.

g) Finally, compare, in the L_2 norm and the infinity norm, each one of the velocity fields obtained at time $t_n = 0.05, 0.1, \cdots 2$ with the velocity fields obtained directly from the exact solution (2), respectively. Make comments about these results.

Final Recommendations

i) Read carefully all the items and try to give answer to all of them. You should also hand in your codes. Make enough comments for a reader to understand the logic of each algorithm in your codes.

ii) I have carefully worked out all the parts of this exam myself. However, I may have overlooked something in the proposed problems, if you think so modify the question and provide an answer to the modified question. In such event, clearly document the error. Points will be deducted if the question in its original form is correct.

iii) I think this test is reasonable and doable in a week. However, you will need TO START EARLY.

iv) THIS IS AN INDIVIDUAL PROJECT! You can use any resource available to you except your classmates. Make an effort to write your own algorithm and obtain your very own answer. Two individuals very rarely write the same code (programming is very personal).