CHAPTER 10 Numerical Solutions of Nonlinear Systems of Equations.

Consider the nonlinear system

\# 5) $$\begin{cases} x_1^2 - 10x_1 + x_2^2 + 8 = 0 & (1) \\ x_1 x_2^2 + x_1 - 10x_2 + 8 = 0 & (2) \end{cases}$$

If $$f_1(x_1, x_2) = x_1^2 - 10x_1 + x_2^2 + 8$$
$$f_2(x_1, x_2) = x_1 x_2^2 + x_1 - 10x_2 + 8$$

and $$\vec{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x_1, x_2) \longrightarrow \vec{F}(x_1, x_2) = (f_1(x_1,x_2), f_2(x_1,x_2))$$

Then, (1)-(2) can be written as

$$\boxed{\vec{F}(x_1, x_2) = \vec{0}} \qquad (3)$$

The fixed point technique for finding roots of (1)-(2) consists of defining an equivalent fixed point problem for (1)-(2).

There are many ways to do this, but not all will be convenient. In this particular case, one possibility is to write (1)-(2) as

$$x_1 = \frac{x_1^2 + x_2^2 + 8}{10} = g_1(x_1, x_2) \quad (4)$$

$$x_2 = \frac{x_1 x_2^2 + x_1 + 8}{10} = g_2(x_1, x_2) \quad (5)$$

or

$$\boxed{\vec{x} = \vec{G}(\vec{x}) = \left(g_1(x_1,x_2),\, g_2(x_1,x_2)\right)} \quad (4)$$

Obviously, a $\underbrace{\text{fixed point of (4)}}$ will be a root of (3) (or (1)-(2)).

$$\downarrow$$

$$\vec{G}(\vec{p}) = \vec{p}.$$

There is an important theorem to determine conditions on $\vec{G}$ in order for this function

① To have a fixed point $\vec{p}$.

② For this point $\vec{p}$ to be unique.

③ To determine if the functional iteration

$$\vec{x}^{(k)} = \vec{G}(\vec{x}^{(k-1)}), \quad k = 1, 2, \ldots$$

Converges to $\vec{p}$ as $k \to \infty$., or $\vec{x}^{(k)} = \vec{G}(\vec{x}^{(k-1)}) \xrightarrow[k\to\infty]{} \vec{p}$.

Also, ④ To determine rate of convergence of functional iteration $\vec{x}^{(k)} = \vec{G}(\vec{x}^{(k-1)})$ to $\vec{p}$.

Theorem.-

① $D \equiv \{(x_1, x_2, \ldots, x_n) \mid a_i \le x_i \le b_i,\ a_i, b_i \in \mathbb{R},\ i = 1, 2, \ldots n\}$

② $\vec{G}: D \subset \mathbb{R}^n \to \mathbb{R}^n$ is conts. $\vec{G}(\vec{x}) = (g_1(\vec{x}), \ldots, g_n(\vec{x}))$

$g_i : D \subset \mathbb{R}^n \to \mathbb{R}$.

③ $\vec{G}$ is onto D. It means

$$\boxed{\vec{G}(\vec{x}) \in D, \text{ for any } \vec{x} \in D.}$$

④ All components $g_i(\vec{x})$ $(i = 1, \ldots, n)$ of $\vec{G}$ have conts part. derivatives.

⑤ There exists $0 < K < 1$ such that

$$\boxed{\left| \frac{\partial g_i}{\partial x_j}(\vec{x}) \right| \le \frac{K}{n}, \quad \vec{x} \in D}$$

For all components $g_i(\vec{x})$ $i = 1, \ldots, n$ and for each variable x_j, $j = 1, \ldots, n$.

Then,

① There is a fixed point $\vec{p}$ for $\vec{G}(\vec{x})$.

② The fixed point $\vec{p}$ is unique.

③ The sequence

$$\boxed{\vec{X}^{(k)} = \vec{G}(\vec{X}^{(k-1)}) \xrightarrow[k\to\infty]{} \vec{p}.}$$

for any initial guess $\vec{X}^{(0)}$.

④ The rate of convergence is given by

$$\boxed{\|\vec{X}^{(k)} - \vec{p}\|_\infty \le \frac{K^k}{1-K} \|\vec{X}^{(1)} - \vec{X}^{(0)}\|_\infty}$$

– Comment on theorem.

In our particular example

① $D = \{ (x_1, x_2) : \ 0 \le x_1, x_2 \le 1.5 \}$

② $g_1(x_1, x_2)$ and $g_2(x_1, x_2)$ are conts on D.

③
$$\frac{8}{10} \le g_1(x_1,x_2) = \frac{x_1^2 + x_2^2 + 8}{10} \le 1.25$$

$$\frac{8}{10} \le g_2(x_1,x_2) = \frac{x_1 x_2^2 + x_1 + 8}{10} \le 1.2875$$

Therefore, $G(\vec{x}) \in [0, 1.5]$ and $\vec{G}$ is onto D.

④

$$\frac{\partial g_1}{\partial x_1} = \frac{2x_1}{10} \Rightarrow \left|\frac{\partial g_1}{\partial x_1}\right| \le \frac{3}{10} \le \frac{9}{20}$$

$$\frac{\partial g_1}{\partial x_2} = \frac{2x_2}{10} \Rightarrow \left|\frac{\partial g_1}{\partial x_2}\right| \le \frac{3}{10} \le \frac{9}{20}$$

$$\frac{\partial g_2}{\partial x_1} = \frac{x_2^2+1}{10} \Rightarrow \left|\frac{\partial g_2}{\partial x_1}\right| \le \frac{3.5}{10} \le \frac{9}{20}$$

$$\frac{\partial g_2}{\partial x_2} = \frac{2x_1x_2}{10} \Rightarrow \left|\frac{\partial g_2}{\partial x_2}\right| \le \frac{4.5}{10} = \frac{9}{20}$$

Therefore

$$\left|\frac{\partial g_i}{\partial x_j}\right| \le \frac{9}{20} = \overbrace{\left(\frac{9}{10}\right)}^{\le K<1}\frac{1}{2}$$

All hypothesis are satisfied then the functional iteration

$$\vec{G}(\vec{x}^{(k-1)}) = \vec{x}^{(k)}$$

Converges to the unique fixed point $\vec{p}$ of $\vec{G}(\vec{x})$.

$$x_1^{(k)} = \frac{\left(x_1^{(k-1)}\right)^2 + \left(x_2^{(k-1)}\right)^2 + 8}{10}$$

$$x_2^{(k)} = \frac{x_1^{(k-1)}\left(x_2^{(k-1)}\right)^2 + x_1^{(k-1)} + 8}{10}$$

Accelarating convergence (Gauss-Seidel technique)

Use latest estimate $x_1^{(k)}, x_2^{(k)}, \dots, x_{i-1}^{(k)}$ to compute $x_i^{(k)}$ instead of $x_1^{(k-1)}, \dots, x_{i-1}^{(k-1)}$.

In our case,

$$x_1^{(k)} = \frac{\left(x_1^{(k-1)}\right)^2 + \left(x_2^{(k-1)}\right)^2 + 8}{10}$$

$$x_2^{(k)} = \frac{x_1^{(k)}\left(x_2^{(k-1)}\right)^2 + x_1^{(k)} + 8}{10}$$

Numerical Results are

FP: $\vec{x}^{(0)} = (0,0)$, $\vec{x}^{(13)} = (0.9999973, 0.9999973)$

TOL $= 10^{-5}$.

FP + G-S: $\vec{x}^{(0)} = (0,0)$, $\vec{x}^{(11)} = (0.9999984, 0.9999991)$