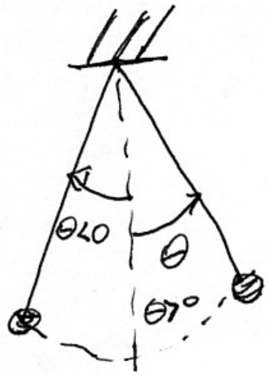
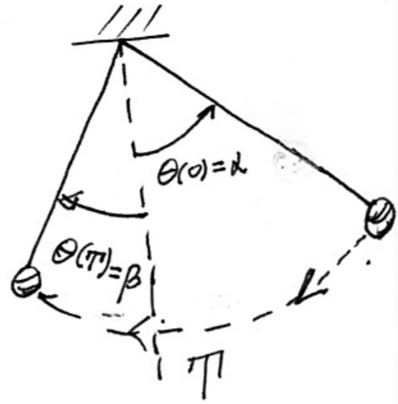
CHAP 11: BVP for ODEs.

Example 1.- Pendulum



$$\begin{cases} \theta''(t) = -\sin(\theta(t)) \\ \theta(0) = \alpha, \quad \theta(T) = \beta \end{cases}$$

Nonlinear



Example 2.- (Book) Deflection of a beam

→ strees ends

→ intensity of uniform load

→ length of beam.

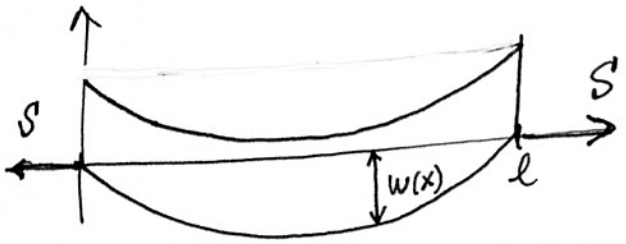
$$\frac{d^2 w}{dx^2} = \frac{S}{EI} w(x) + \frac{q x}{2EI}(x-\ell)$$

Elas. Modulus

Central moment inertia

Linear

$$w(0) = 0, \quad w(\ell) = 0$$



In general, BVP for 2^{nd} order ODEs

$$\begin{cases} y'' = f(x, y, y'), & a < x < b \\ y(a) = \alpha, \quad y(b) = \beta \end{cases}$$

Shooting Method definition. Linear case

We want to solve the BVP

$$\begin{cases} y'' = p(x)y' + q(x)y + r(x), & a \le x \le b, \quad (1) \\ y(a) = \alpha, \quad y(b) = \beta & (2) \end{cases}$$

For this, we consider two other linear problems, but only IVPs.

$$(IVP_1) \quad \begin{cases} y'' = p(x)y' + q(x)y + r(x), & a < x < b, \quad (3) \\ y(a) = \alpha, \quad y'(a) = 0 & (4) \end{cases}$$

$$(IVP_2) \quad \begin{cases} y'' = p(x)y' + q(x)y, & a < x < b, \quad (5) \\ y(a) = 0, \quad y'(a) = 1 & (6) \end{cases}$$

If ① $p(x)$, $q(x)$, and $r(x)$ are conts. on $[a,b]$.

② $q(x) > 0$, on $[a,b]$.

Then, BVP (1)-(2) has a unique soln: $y(x)$

Also, IVP_1 and IVP_2 have unique solns: $y_1(x)$ and $y_2(x)$.

A linear combination of $y_1(x)$ and $y_2(x)$:

$$y(x) \equiv y_1(x) + B\, y_2(x), \qquad (3.1)$$

is also a soln. of (1). In fact,

$$y'' = (y_1 + B y_2)'' = y_1'' + B y_2'' = p(x)\, y_1' + q(x) y_1 + r(x) + B\big(p(x)\, y_2' + q(x)\, y_2\big)$$

$$= p(x)\big(y_1' + B y_2'\big) + q(x)\big(y_1 + B y_2\big) + r(x)$$

$$= p(x)\, y'(x) + q(x)\, y(x) + r(x) \quad \checkmark$$

Now, we will determine B to satisfy BCs (2).

First, we want to satisfy: $y(a) = \alpha$

This condition is trivially satisfy for any B, since

$$y(a) = y_1(a) + B y_2(a) = \alpha + B(0) = \alpha.$$

Sencondly, we would like to satisfy: $y(b) = \beta$

This requires

$$\beta = y(b) = y_1(b) + B\, y_2(b) \Longrightarrow \boxed{B = \frac{\beta - y_1(b)}{y_2(b)}}$$

Therefore,

$$\boxed{y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x)} \qquad (4.1)$$

Satisfies BVP (1)-(2). This is the unique soln.

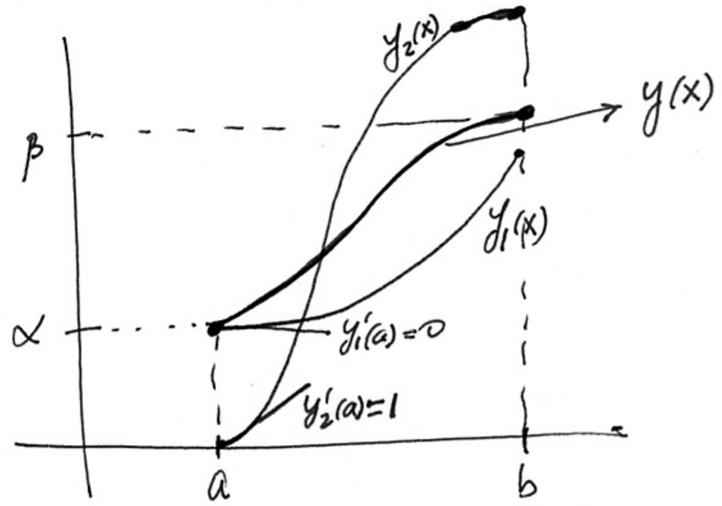
Notice that

$$y'(x) = y_1'(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2'(x)$$

$$y'(a) = \overbrace{y_1'(a)}^{=0} + \frac{\beta - y_1(b)}{y_2(b)} \overbrace{y_2'(a)}^{=1} = \frac{\beta - y_1(b)}{y_2(b)}$$

I don't see why $y'(a) = 1$ in (6). Actually any value $y'(a) = c^{\neq 0}$ would work.

The three solns. can be graphed as



Rmk: Another justification for this approach is that $y_1(x)$ and $y_2(x)$ are

Equation (4.1) suggests to approximate $y(x)$ by first solving approximately IVPs for $y_1(x)$ and $y_2(x)$.

One way (approach) would be to use R-K 4th order for IVPs (3)-(4) and (5)-(6). This is done in book (B-F.)

In order to do this, both IVPs should be reduced to 2x2 systems of ODE.

IVP$_1$) $u_1 = y$, $u_2 = y'$

Then
$$\begin{cases} u_1' = u_2 \\ u_2' = p(x)u_2 + q(x)u_1 + r(x) \end{cases}$$

IVP$_2$) $v_1 = y$, $v_2 = y'$

$$\begin{cases} v_1' = v_2 \\ v_2' = p(x)v_2 + q(x)v_1 \end{cases}$$

R-K 4th.

$w_0 = \alpha$

$k_1 = hf(t_i, w_i)$

$k_2 = hf(t_{i+1/2}, w_i + \frac{1}{2}k_1)$

$k_3 = hf(t_{i+1/2}, w_i + \frac{1}{2}k_2)$

$k_4 = hf(t_{i+1}, w_i + k_3)$

$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

4 first terms in Alg 11.1

$$\boxed{u_1' = u_2 = f_1(x, u_1, u_2)}$$
$$\boxed{u_2' = p(x)u_2 + q(x)u_1 + r(x) = f_2(x, u_1, u_2)}$$

Then,

$$K_{1,1} = h f_1(x, u_1, u_2) = h u_2$$

$$K_{1,2} = h f_2(x, u_1, u_2) = h\left[p(x)u_2 + q(x)u_1 + r(x)\right]$$

$$K_{2,1} = h f_1\left(x + \tfrac{h}{2}, u_1 + \tfrac{1}{2}K_{1,1}, u_2 + \tfrac{1}{2}K_{1,2}\right) = h\left[u_2 + \tfrac{1}{2}K_{1,2}\right]$$

$$K_{2,2} = h f_2\left(x + h/2, u_1 + \tfrac{1}{2}K_{1,1}, u_2 + \tfrac{1}{2}K_{1,2}\right) =$$
$$= p(x + h/2)\left[u_2 + \tfrac{1}{2}K_{1,2}\right] + q(x + h/2)\left(u_1 + \tfrac{1}{2}K_{1,1}\right) + r(x + h/2).$$

$\vdots$