Matrix Stability Analysis

Consider the initial boundary value problem (IBVP)

$$u_t = \sigma u_{xx}, \qquad 0 < x < 1, t > 0 \tag{1}$$
$$u(0,t) = g(t), \qquad u(1,t) = h(t) \tag{2}$$
$$u(x,0) = f(x) \tag{3}$$

Equation (1) can be written as

$$u_t = Lu, \tag{4}$$

where L is a linear differential operator.

We have seen three different numerical schemes to approximate the solution of IBVP (1)-(3). They are

1. **Forward in time–Centered in space**
$$U_i^{n+1} = rU_{i-1}^n + (1-2r)U_i^n + rU_{i+1}^n, \quad i = 1, \ldots m, \tag{5}$$
where $r = \sigma\Delta t/\Delta x^2$. This scheme is $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$. The linear system that results from (5) can be represented by
$$\mathbf{U}^{n+1} = L_\Delta^F \mathbf{U}^n \tag{6}$$

2. **Backward in time–Centered in space**
$$-rU_{i-1}^{n+1} + (1+2r)U_i^{n+1} - rU_{i+1}^{n+1} = U_i^n, \quad i = 1, \ldots m \tag{7}$$
This scheme is $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$ The linear system that results from (7) can be represented by
$$L_\Delta^B \mathbf{U}^{n+1} = \mathbf{U}^n \quad \text{or} \quad \mathbf{U}^{n+1} = (L_\Delta^B)^{-1}\mathbf{U}^n \tag{8}$$

3. **Crank–Nicholson**
$$\frac{-r}{2}U_{i-1}^{n+1} + (1+r)U_i^{n+1} - \frac{r}{2}U_{i+1}^{n+1} = \frac{r}{2}U_{i-1}^n + (1-r)U_i^n + \frac{r}{2}U_{i+1}^n, \quad i = 1, \ldots m \tag{9}$$
This scheme is $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$ The linear system that results from (9) can be represented by
$$L_\Delta^S \mathbf{U}^{n+1} = L_\Delta^G \mathbf{U}^n \quad \text{or} \quad \mathbf{U}^{n+1} = (L_\Delta^S)^{-1} L_\Delta^G \mathbf{U}^n = (L_\Delta^{CN})\mathbf{U}^n \tag{10}$$

0.1 Definition 1: Stability

A linear finite difference method (FDM) of the form

$$\mathbf{U}^{n+1} = L_\Delta \mathbf{U}^n \tag{11}$$

corresponding to an IBVP of (4) (such as (1)-(3)) is stable if there exists $C > 0$, independent of the mesh spacing and the initial data, such that

$$||\mathbf{U}^n|| \leq C||\mathbf{U}^0||, \qquad n \to \infty, \quad \Delta t \to 0, \quad \Delta x \to 0, \quad n\Delta t \leq T \tag{12}$$

0.2 Theorem 1: Equivalent Condition

The FDM (11) is stable if and only if there exists a constant $C > 0$ independent of Δx and Δt such that

$$||(L_\Delta)^n|| \leq C, \qquad n \to \infty, \quad \Delta t \to 0, \quad \Delta x \to 0, \quad n\Delta t \leq T \tag{13}$$

Remark: Notice that C may be greater than 1.

0.3 Corollary 1: Practical Condition

If the discrete operator L_Δ of the FDM (11) satisfies

$$||L_\Delta|| \leq 1,$$

then the FDM (11) is stable.

Remark: Apply this condition to the explicit FDM FT-CS using the infinity norm.

0.4 Corollary 2: More General Condition

If there is is a $c > 0$ independent of Δx and Δt such that the discrete operator L_Δ of the FDM (11) satisfies

$$||L_\Delta|| \leq 1 + c\Delta t,$$

then the FDM (11) is stable.

0.5 Definition 2: Spectral Radius

The spectral radius $\rho(L_\Delta)$ of the FDM matrix L_Δ is the absolute value of its largest eigenvalue. Assuming that λ_i, $i = 1, \ldots N$ are the eigenvalues of L_Δ, then

$$\rho(L_\Delta) = \max_{1 \leq i \leq N} |\lambda_i|$$

0.6 Theorem 2: Relationship Between Spectral Radius and Norm of L_Δ

If $\rho(L_\Delta)$ and L_Δ are the spectral radius and the vector-induced norm of L_Δ then,

$$\rho(L_\Delta) \leq ||L_\Delta||$$

0.7 Corollary 3: Necessary Condition

The condition

$$\rho^n(L_\Delta) \leq C,$$

for a constant $C > 0$ independent of Δx and Δt is a necessary condition for the stability of the FDM (11).

0.8 Corollary 4: A More Practical Condition (special matrices)

If L_Δ of the FDM (11) is symmetric or similar to a symmetric matrix, then

$$\rho(L_\Delta) \leq 1,$$

for any Δx and Δt, is also a sufficient condition for stability in the Euclidean norm.

Remark: Apply this condition to show stability of FT-CS and BT-CS FDM for IBVP (1)-(3) with homogeneous boundary conditions.

Why do we want to prove stability for FDM such as (11) approximating certain PDE problems modelled by (4)?

The answer to this question is found in the next theorem

0.9 Theorem 3: Lax-Equivalence Theorem

A consistent linear FDM such as (11) is convergent if and only if it is stable.

In many problems of practical interest, we would like to study stability when $t \to \infty$. To analyze stability for these problems, we need an alternative stability definition.

0.10 Definition 3: Absolute Stability

A FDM such as (11) is absolutely stable for a given mesh (of size Δx and Δt) if

$$||\mathbf{U}^n|| \leq ||\mathbf{U}^0||, \qquad n > 0 \tag{14}$$

0.11 Definition 4: Unconditional Stability

A FDM such as (11) is unconditionally stable if it is absolutely stable for all choices of mesh spacing Δx and Δt.