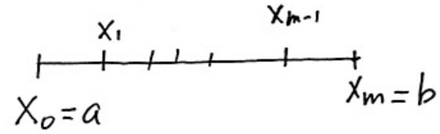
Stability Analysis



I) FT-CS for heat equation with homogeneous BCs.

$$\vec{U}^{n+1} = L_\Delta^F \vec{U}^n \tag{0}$$

where

$$L_\Delta^F = \begin{bmatrix} 1-2r & r & 0 & 0 & \cdots & 0 \\ r & 1-2r & r & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & r & 1-2r & & r \\ 0 & \cdots & & & r & 1-2r \end{bmatrix}_{(m-1)\times(m-1)} . \tag{1}$$

$$r \equiv \frac{\sigma \Delta t}{(\Delta x)^2}$$

Two options:

a) Using infinity norm $\|\cdot\|_\infty$

If $A = (a_{ij})_{N\times N}$ then

$$\|A\|_\infty \stackrel{def}{=} \max_{1\le i\le N} \left(\sum_{j=1}^{N} |a_{ij}| \right)$$

For (1) $\quad \|L_\Delta^F\|_\infty = |r| + |1-2r| + |r| \stackrel{r>0}{=} 2r + |1-2r|$

Then, if $r \le 1/2 \Rightarrow 1-2r \geq 0$

Thus, $\quad \|L_\Delta^F\|_\infty = 2r + 1 - 2r = 1$

According to corollary 1, the FDM (0) is stable.

Why is this not true if $r > 1/2$?

Using L_2 norm (Spectral radius)

b) Matrices of the form:

$$A = \begin{bmatrix} d_0 & d_1 & 0 & 0 & \cdots & 0 \\ d_1 & d_0 & d_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & d_1 & d_0 & d_1 \\ 0 & \cdots & \cdots & 0 & d_1 & d_0 \end{bmatrix}_{(m-1)\times(m-1)}$$

Symmetric and Tridiagonal (2.1)

have eigenvalues: $\boxed{\lambda_p = d_0 + 2d_1 \cos\left(\frac{p\pi}{m}\right) \quad p = 1, 2, \ldots m}$ (2.2)

Therefore, matrix (1) have eigenvalues:

$$\lambda_p = 1 - 2r + 2r\cos\left(\frac{p\pi}{m}\right) = 1 - 2r\left(1 - \cos\left(\frac{p\pi}{m}\right)\right)$$

$p = 1, 2, .., m-1$

But, $\quad 1 - \cos\left(\frac{p\pi}{m}\right) = 2\sin^2\left(\frac{p\pi}{2m}\right)$

Then, $\quad \boxed{\lambda_p = 1 - 4r\sin^2\left(\frac{p\pi}{2m}\right),} \qquad p = 1, 2, ., m-1$

According to corollary 4, we only need to find Conditions on r for $\underset{\substack{\text{for all } p \\ \text{for all } m}}{|\lambda_p| \leq 1} \Rightarrow \rho(A) \leq 1$.

or

$$-1 \leq 1 - 4r\sin^2\left(\frac{p\pi}{2m}\right) \leq 1$$

or $\quad -1 \leq 4r\sin^2\left(\frac{p\pi}{2m}\right) - 1 \leq 1$

$$0 \le 4r\sin^2\left(\frac{p\pi}{2m}\right) \le 2$$

$$\Rightarrow \qquad r \le 1/2. \iff \frac{\sigma \Delta t}{\Delta x^2} \le 1/2 \Rightarrow \Delta t \le \frac{\Delta x^2}{2\sigma}$$

too restrictive!

II) BT-CS for Heat cond with homog. BCs.

$$L_\Delta^B \vec{U}^{n+1} = \vec{U}^n \qquad (3.1)$$

where

$$L_\Delta^B = \begin{bmatrix} 1+2r & r & 0 & 0 & \cdots & 0 \\ r & 1+2r & r & 0 & \cdots & 0 \\ 0 & & \ddots & & & \vdots \\ \vdots & & & r & 1+2r & r \\ 0 & \cdots & & & r & 1+2r \end{bmatrix} \qquad (3.2)$$

Equ. (3.1) implies

$$\vec{U}^{n+1} = \left(L_\Delta^B\right)^{-1} \vec{U}^n$$

Since L_Δ^B is Symmetric $\Rightarrow$ $\left(L_\Delta^B\right)^{-1}$ is also Symmetric.

Therefore, we only need to find conditions on r, such that

$$\rho\left(\left(L_\Delta^B\right)^{-1}\right) \le 1$$

to guarantee stability in the Euclidean norm. (according to Corollary 4)

Matrix (3.2) is of the form of (3.1), then

$$\lambda_p = 1+2r+2r\cos\left(\frac{p\pi}{m}\right) \qquad p=1,2,..,m-1$$

or $$\lambda_p = 1+2r\left(1+\cos\left(\frac{p\pi}{m}\right)\right) =$$

$$= 1+2r\left(2\cdot\cos^2\left(\frac{p\pi}{2m}\right)\right)$$

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$

$\cos(2\alpha) = 2\cos^2\alpha - 1$

$(1+\cos 2\alpha) = 2\cos^2\alpha.$

or $$\lambda_p = 1+4r\cos^2\left(\frac{p\pi}{2m}\right), \qquad r = \frac{\sigma\Delta t}{(\Delta x)^2} > 0$$

Then, $\lambda_p \geq 1$, for all $p=1,..,m-1$

including the smallest eigenvalue in magnitude λ_{p^*}

Therefore, $$\rho\left(\left(L_\Delta^B\right)^{-1}\right) = \frac{1}{\text{Smallest eigenv. of } L_\Delta^B} = \frac{1}{|\lambda_{p^*}|} \leq 1$$

for any value of "r".

As a consequence, BT-CS FDM is unconditionally stable, independent of the choice of Δt and Δx.

Example 1 Consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \quad 0 \le t,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 < t,$$

and initial conditions

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1.$$

The solution to this problem is

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x).$$

The solution at $t = 0.5$ will be approximated using the Forward-Difference method, first with $h = 0.1$, $k = 0.0005$, and $\lambda = 0.05$, and then with $h = 0.1$, $k = 0.01$, and $\lambda = 1$. The results are presented in Table 12.3. ■

$\lambda = \frac{5\times10^{-4}}{(10^{-1})^2} = 0.05 < 1/2$ $\qquad$ $\lambda = \frac{10^{-2}}{(10^{-1})^2} = 1 > 1/2$

12.3

$u(x_i, 0.5)$	$w_{i,1000}$ $k = 0.0005$	$\lvert u(x_i, 0.5) - w_{i,1000}\rvert$	$w_{i,50}$ $k = 0.01$	$\lvert u(x_i, 0.5) - w_{i,50}\rvert$
0	0		0	
0.00222241	0.00228652	6.411×10^{-5}	8.19876×10^{7}	8.199×10^{7}
0.00422728	0.00434922	1.219×10^{-4}	-1.55719×10^{8}	1.557×10^{8}
0.00581836	0.00598619	1.678×10^{-4}	2.13833×10^{8}	2.138×10^{8}
0.00683989	0.00703719	1.973×10^{-4}	-2.50642×10^{8}	2.506×10^{8}
0.00719188	0.00739934	2.075×10^{-4}	2.62685×10^{8}	2.627×10^{8}
0.00683989	0.00703719	1.973×10^{-4}	-2.49015×10^{8}	2.490×10^{8}
0.00581836	0.00598619	1.678×10^{-4}	2.11200×10^{8}	2.112×10^{8}
0.00422728	0.00434922	1.219×10^{-4}	-1.53086×10^{8}	1.531×10^{8}
0.00222241	0.00228652	6.511×10^{-5}	8.03604×10^{7}	8.036×10^{7}
0	0		0	

Backward Time Centered in Space

subject to the constraints

$$u(0,t) = u(1,t) = 0, \quad 0 < t, \quad u(x,0) = \sin \pi x, \quad 0 \le x \le 1,$$

considered in Example 1. To demonstrate the unconditional stability of the Backward-Difference method, we again compare $w_{i,50}$ to $u(x_i, 0.5)$, where $i = 0, 1, \ldots, 10$.

The results listed in Table 12.4 have the same values of h and k as those in the fifth and sixth columns of Table 12.3, which illustrates the stability of this method. ∎

$\lambda = \frac{10^{-2}}{(10^{-1})^2} = 1$, $h = 0.1$, $k = 0.01$

Table 12.4

x_i	$w_{i,50}$	$u(x_i, 0.5)$	$\lvert w_{i,50} - u(x_i, 0.5)\rvert$
0.0	0	0	
0.1	0.00289802	0.00222241	6.756×10^{-4}
0.2	0.00551236	0.00422728	1.285×10^{-3}
0.3	0.00758711	0.00581836	1.769×10^{-3}
0.4	0.00891918	0.00683989	2.079×10^{-3}
0.5	0.00937818	0.00719188	2.186×10^{-3}
0.6	0.00891918	0.00683989	2.079×10^{-3}
0.7	0.00758711	0.00581836	1.769×10^{-3}
0.8	0.00551236	0.00422728	1.285×10^{-3}
0.9	0.00289802	0.00222241	6.756×10^{-4}
1.0	0	0	

Crank-Nicholson

Table 12.5

x_i	$w_{i,50}$	$u(x_i, 0.5)$	$\lvert w_{i,50} - u(x_i, 0.5)\rvert$
0.0	0	0	
0.1	0.00230512	0.00222241	8.271×10^{-5}
0.2	0.00438461	0.00422728	1.573×10^{-4}
0.3	0.00603489	0.00581836	2.165×10^{-4}
0.4	0.00709444	0.00683989	2.546×10^{-4}
0.5	0.00745954	0.00719188	2.677×10^{-4}
0.6	0.00709444	0.00683989	2.546×10^{-4}
0.7	0.00603489	0.00581836	2.165×10^{-4}
0.8	0.00438461	0.00422728	1.573×10^{-4}
0.9	0.00230512	0.00222241	8.271×10^{-5}
1.0	0	0	