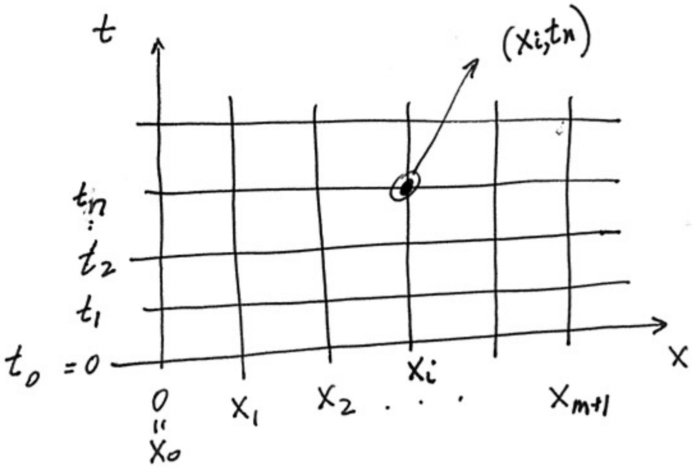
CHAPTER 9 Diffusion Equations and Parabolic Problems.

Consider IBVP:

$$\begin{cases} u_t = \sigma u_{xx}, & 0<x<1, \quad \sigma>0. & (1)\\ u(x,0) = \eta(x) & & (2)\\ u(0,t) = g_0(t), & t>0 & (3)\\ u(1,t) = g_1(t), & t>0 & (4)\end{cases}$$

Discretization : Grid



$x_i = ih$

$t_n = nk$

$h = \Delta x$

$k = \Delta t$

$U_i^n \simeq u(x_i, t_n)$

FT-CS numerical method

(I) Approx. of $(u_t)_i^n$ (forward difference)

$$u(x_i, t_{n+1}) = u_i^{n+1} = u_i^n + k(u_t)_i^n + \frac{k^2}{2}(u_{tt})_i^{n+\theta}$$

$$\Rightarrow \quad (u_t)_i^n = \frac{u_i^{n+1} - u_i^n}{k} - \frac{k}{2}(u_{tt})_i^{n+\theta} \qquad (5)$$

$0<\theta<1$

Similarly, centered approx. of $(u_{xx})_i^n$

(II) $$(u_{xx})_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} - \frac{h^2}{12}(u_{xxxx})_{i+\xi}^n \quad (6)$$

$0<\xi<1$

Subst. (5)-(6) into (1)

$$\frac{u_i^{n+1} - u_i^n}{k} - \frac{k}{2}(u_{tt})_i^{n+\theta} = \sigma\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} - \frac{h^2}{12}(u_{xxxx})_{i+\xi}^n\right]$$

$i=1,2,.,m$

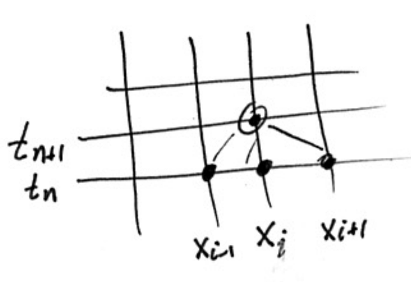
Neglecting discretization errors, it leads to

$$\boxed{\frac{v_i^{n+1} - v_i^n}{k} = \frac{\sigma}{h^2}\left(v_{i+1}^n - 2v_i^n + v_{i-1}^n\right)} \quad (6.1)$$

Explicit method

$$\boxed{v_i^{n+1} = v_i^n + \frac{\sigma k}{h^2}\left(v_{i+1}^n - 2v_i^n + v_{i-1}^n\right),} \quad i=1,2,\dots m. \quad (7)$$

FT-CS finite difference method for Heat Cond. (1-D).



STENCIL

Explain "marching in time" process.

Definition of Local Truncation Error

If the exact Soln. $u(x,t)$ is Subst. into (6.1)

$$\tau_i^n \equiv \frac{u_i^{n+1} - u_i^n}{k} - \sigma\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}\right) =$$

$$= (u_t)_i^n + \frac{k}{2}(u_{tt})_i^{n+\theta} - \sigma\left((u_{xx})_i^n + \frac{h^2}{12}(u_{4x})_{i+\xi}^n\right) \qquad (7.1)$$

$$= \left[\overbrace{(u_t)_i^n - \sigma (u_{xx})_i^n}^{=0}\right] + \frac{k}{2}(u_{tt})_i^{n+\theta} - \frac{\sigma h^2}{12}(u_{4x})_{i+\xi}^n$$

→ This expression is defined as the local truncation error of equation (1) using the FT-CS FDM.

In more general terms, by defining.

$$L_\Delta v_i^n \equiv \frac{v_i^{n+1} - v_i^n}{k} - \frac{\sigma}{h^2}\left(v_{i+1}^n - 2v_i^n + v_{i-1}^n\right) \quad \text{(Discrete Operator)}$$

and $L u(x,t) \equiv u_t - \sigma u_{xx}$. / Equ. (1) → $L u(x,t) = 0$ (Continuous Operator)

From (7.1), we arrive to

$$\tau_i^n = L_\Delta u_i^n - \overbrace{L u(x_i, t_n)}^{=0} = \frac{k}{2}(u_{tt})_i^{n+\theta} - \frac{\sigma h^2}{12}(u_{4x})_{i+\xi}^n$$

$$= \mathcal{O}(k) + \mathcal{O}(h^2).$$

Def.– If L is the conts diff. operator defining the PDE: $L u(x,t) = 0$ and L_Δ is a discrete operator acting on v_i^n, $L_\Delta v_i^n = 0$. Then, the Local truncation error τ_i^n is defined as

$$\boxed{\tau_i^n \equiv L_\Delta u_i^n - L u(x_i, t_n)}$$

where $u(x,t)$ is an exact soln. of $Lu(x,t) = 0$.

Equ. (7) Can also be written as

$$\boxed{v_i^{n+1} = r\,v_{i-1}^{n} + (1-2r)\,v_i^{n} + r\,v_{i+1}^{n}}\,, \quad \text{where } r \equiv \frac{\sigma h}{K^2}. \quad (8)$$

$$i = 1, .., m.$$

Implicit schemes

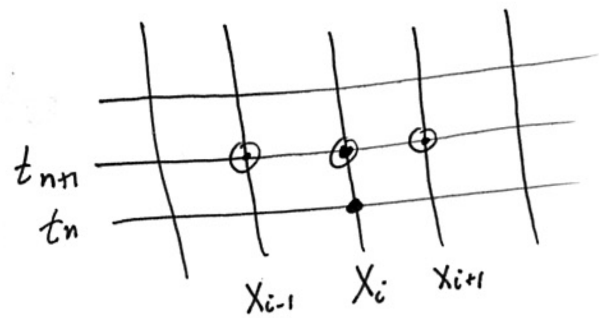
a) BT-CS. This one is centered around (x_i, t_{n+1})

$$\frac{v_i^{n+1} - v_i^{n}}{K} = \sigma\,\frac{v_{i+1}^{n+1} - 2v_i^{n+1} + v_{i-1}^{n+1}}{h^2}$$

It reduces to

$$\boxed{-r\,v_{i-1}^{n+1} + (1+2r)\,v_i^{n+1} - r\,v_{i+1}^{n+1} = v_i^{n}, \quad i = 1, .., m} \qquad r = \frac{\sigma K}{h^2} \quad (9)$$

Stencil



Compare computation process betw. (8) and (9).

To obtain approximations of $u(x_i, t_n)$ $\quad i = 1, .., m$, $\quad n = 1, \dots N$

we need to solve a linear syst. for the unkns: $U_1^n, U_2^n, .., U_m^n$; at each time t_n. In fact, Equ. (9) can be written as

$$A \vec{U}^{n+1} = \vec{F}^n \qquad (10)$$

$$A = \begin{bmatrix} 1+2r & -r & 0 & 0 & \cdots & 0 \\ -r & 1+2r & -r & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & -r & 1+2r \end{bmatrix}$$

$$\vec{U}^{n+1} = \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_m^{n+1} \end{bmatrix}, \quad \vec{F}^n = \begin{bmatrix} U_1^n + r \overbrace{U_0^{n+1}}^{= g_0(t_{n+1})} \\ U_2^n \\ \vdots \\ U_m^n \\ U_m^n + r \overbrace{U_{m+1}^{n+1}}^{= g_1(t_{n+1})} \end{bmatrix}$$

Implicit Schemes

b) Crank-Nicholson method

It's similar to BT-CS, but the forcing term is more complicated.

$$\boxed{\begin{aligned} -rU_{i-1}^{n+1} + (1+2r)U_i^{n+1} - rU_{i+1}^{n+1} &= \\ = rU_{i-1}^{n} + (1-2r)U_i^{n} + rU_{i+1}^{n} \\ i=1,.,m \end{aligned}} \quad (11)$$

Derivation:

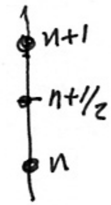
We approximate : $u_t = \sigma u_{xx}$ at the point $(x_i, t_{n+1/2})$ using centered difference in both time and space with time step size $\frac{\Delta t}{2}$.

$$(u_t)_i^{n+1/2} = \sigma (u_{xx})_i^{n+1/2}$$

CT-CS with time step $\frac{\Delta t}{2} = k/2$ and $\Delta x = h$. at $(x_i, t_{n+1/2})$

$$\frac{U_i^{n+1} - U_i^{n}}{2\left(\frac{k}{2}\right)} = \sigma \frac{U_{i+1}^{n+1/2} - 2U_i^{n+1/2} + U_{i-1}^{n+1/2}}{h^2}.$$

Using average in time



$$\frac{U_i^{n+1} - U_i^{n}}{k} = \sigma \frac{\frac{U_{i+1}^{n+1} + U_{i+1}^{n}}{2} - 2\,\frac{U_i^{n+1} + U_i^{n}}{2} + \frac{U_{i-1}^{n+1} + U_{i-1}^{n}}{2}}{h^2}$$

or

$$U_i^{n+1} - U_i^{n} = \frac{\overbrace{\sigma \frac{k}{h^2}}^{\equiv r}}{2} \left[U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1} + U_{i+1}^{n} - 2U_i^{n} + U_{i-1}^{n} \right]$$

$$\boxed{-\frac{r}{2} U_{i-1}^{n+1} + (1+r) U_i^{n+1} - \frac{r}{2} U_{i-1}^{n+1} = \frac{r}{2} U_{i-1}^{n} + (1-r) U_i^{n} + \frac{r}{2} U_{i+1}^{n}} \qquad (12)$$