Thm 5.20

Proof:- (the method is stable) (Exercise 1 in book)

Let $\{u_i\}_0^N$ and $\{v_i\}_0^N$ be two discrete solns. of (4) such that $|u_0 - v_0| < \varepsilon$.

Then, for $h > 0$ (fixed)

$$|u_i - v_i| = |u_{i-1} - v_{i-1} + h[\phi(t_{i-1}, u_{i-1}, h) - \phi(t_{i-1}, v_{i-1}, h)]|$$

$$\le |u_{i-1} - v_{i-1}| + hL|u_{i-1} - v_{i-1}| = (1 + hL)|u_{i-1} - v_{i-1}| \le$$

$$\le (1+hL)^2 |u_{i-2} - v_{i-2}| \le \ldots \le (1+hL)^i |u_0 - v_0|$$

$$< (1+hL)^i \varepsilon \equiv K\varepsilon.$$

Example 1.- Modified Euler is stable if $f(t,w)$ satisfies a Lip. cond. and if $f$ is conts on $D$. $D = \{(t,w) : t \in [a,b], w \in (-\infty, \infty)\}$

$$w_0 = \alpha, \qquad w_{i+1} = w_i + \frac{h}{2}\Big[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))\Big]$$

or

$$w_{i+1} = w_i + h\,\phi(t_i, w_i, h)$$

Proof.-
① We will prove $\phi$ satisfies a Lipsch. cond.
② $\phi$ is conts.
Then all conclusions of thm 5.20 holds.

① <u>Lip. cond.</u> for Modified – Euler.

$$|\phi(t,w,h) - \phi(t,\bar{w},h)| = \frac{1}{2}\Big| f(t,w) - f(t,\bar{w}) +$$

$$f(t+h, w+h f(t,w)) - f(t+h, \bar{w}+h f(t,\bar{w}))\Big| \le$$

$$\le \frac{1}{2}L|w-\bar{w}| + \frac{1}{2}L\,|w+h f(t,w) - \bar{w} - h f(t,\bar{w})| \le$$

$$\le \frac{1}{2}L|w-\bar{w}| + \frac{1}{2}L\,|w-\bar{w}| + \frac{h}{2}L\,|f(t,w) - f(t,\bar{w})| \le$$

$$L|w-\bar{w}| + \frac{h}{2}L^2|w-\bar{w}| = \left(L + \frac{h}{2}L^2\right)|w-\bar{w}|$$

or $$|\phi(t,w,h) - \phi(t,\bar{w},h)| \le \hat{L}\,|w-\bar{w}|$$

where $$\left(L+\frac{h}{2}L^2\right) \le \underbrace{\left(L + \frac{h_0}{2}L^2\right)}_{\equiv \hat{L}}, \text{ for all } h \le h_0$$

② $$\phi(t,w,h) = \frac{1}{2}\big[f(t,w) + f(t+h, w+h f(t,w))\big]$$ is clearly conts ~~if~~ $f$ is conts. $\Rightarrow$ <u>Modif. Euler is stable</u> $\Rightarrow$ M.E. is converg.

– Notice that

$$\phi(t,w,0) = \frac{1}{2}f(t,w) + \frac{1}{2}f(t,w) = f(t,w).$$

this implies consistency, since

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i, h) \xrightarrow[h\to 0]{} \underbrace{f(t_i,y_i)}_{\lim_{h\to 0}\frac{y_{i+1}-y_i}{h} = f(t_i,y_i)} - \phi(t_i,y_i,0) = 0$$

Comment on: i) Convergence, ii) $O(h^2)$ due to (iii)

# Multistep Methods. Stability and Convergence.

Let's start with a particular case A-B 4th order

$m-1=3$

$$w_0=\alpha,\quad w_1=\alpha_1,\ \ldots,\ w_3=\alpha_3.$$

$$w_{i+1}=w_i+\frac{h}{24}\left[55f(t_i,w_i)-59f(t_{i-1},w_{i-1})+37f(t_{i-2},w_{i-2})-9f(t_{i-3},w_{i-3})\right] \quad (3.1)$$

$i=3,4,\ldots,N-1$

or

$$w_{i+1}=w_i+hF(t_i,h,w_i,w_{i-1},\ldots,w_{i-3}) \quad (3.2)$$

**Def.** — We say that $F$ satisfies a Lipschitz cond. with respect to the seq. $\{w_j\}_0^N$ if there is $L>0$ s.t.

$$|F(t_i,h,v_i,\ldots,v_{i-3})-F(t_i,h,\tilde{v}_i,\ldots,\tilde{v}_{i-3})|\le L\sum_{j=0}^{3}|v_{i-j}-\tilde{v}_{i-j}|$$

for every pair of sequences $\{v_j\}_0^N$, $\{\tilde{v}_j\}_0^N$ and for $i=m-1,m,\ldots,N-1$

We show now that $F$ satisfies a Lipschitz cond. w.r. to $\{w_j\}_0^N$ if $f(t,y)$ satisfies a Lipschitz cond. in "$y$"

In fact,

Exercise #2

using that $f$ is Lipsch.

$$|F(t_i,h,v_i,\ldots,v_{i-3})-F(t_i,h,\tilde{v}_i,\ldots,\tilde{v}_{i-3})|\le\frac{55L}{24}|v_i-\tilde{v}_i|+\frac{59L}{24}|v_{i-1}-\tilde{v}_{i-1}|+\frac{37L}{24}|v_{i-2}-\tilde{v}_{i-2}|+\frac{9L}{24}|v_{i-3}-\tilde{v}_{i-3}|\le$$

$$\le\frac{59}{24}L\sum_{j=0}^{3}|v_{i-j}-\tilde{v}_{i-j}|$$

Exercises:

5.10: 2(a-d), 4-6, 8

5.11: 1, 3, 5 (a) (c), 10.

For a more general concept of Lip. cond. for Multistep methods consider pp. 328.

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Def.- (L.T.E.)

If $w_0 = \alpha,\ w_1 = \alpha_1, \dots, \quad w_{m-1} = \alpha_{m-1}$

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i-(m-1)} +$$

$$+ h F(t_i, h, w_{i+1}, \dots, w_{i-(m-1)}), \qquad i = m-1, \dots, N-1.$$

Then

$$L.T.E \equiv \tau_{i+1}(h) = \frac{y_{i+1} - a_{m-1} y_i - \dots - a_0 y_{i-(m-1)}}{h} -$$

$$- F(t_i, h, y_{i+1}, \dots, y_{i-(m-1)}), \qquad i = m-1, \dots, N-1$$

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Def.- (Consistency)

Two conditions should be satisfied for consistency

i) $\lim_{h \to 0} |\tau_i(h)| = 0, \qquad i = m, m+1, \dots, N$

ii) $\lim_{h \to 0} |\alpha_i - y_i| = 0, \qquad i = 1, 2, \dots, m-1$

They say (book) consistency but I would say convergence of one-step method.

Consider the IVP

$$y'(t) \equiv 0, \quad y(a) = \alpha, \quad \alpha \neq 0 \qquad (5.1)$$

Exact Soln: $y(t) \equiv \alpha$

we now apply general MS method (5) to (5.1)

we will assume that if $f \equiv 0 \Rightarrow F \equiv 0$

Therefore, the MS method reduces to

$$\boxed{w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdots + a_0 w_{i-(m-1)}} \qquad (5.2)$$

and any starting method of one-step will be reduced to

$$w_{i+1} = w_i, \qquad i = 0, .., m-2$$

then $w_0 = \alpha \Rightarrow w_1 = \alpha = w_2 = \cdots = w_{m-2} = w_{m-1}$

and using the MS method

$$w_m = a_{m-1} w_{m-1} + a_{m-2} w_{m-2} + \cdots + a_0 w_0 =$$

$$= \alpha \left( a_{m-1} + a_{m-2} + \cdots + a_0 \right).$$

Since, we want the solution of (5.2) to approx. the soln. of (5.1)

then, $w_m = \alpha$ and $\alpha = \alpha \left( a_{m-1} + a_{m-2} + \cdots + a_0 \right)$

$$\Rightarrow \boxed{1 - a_{m-1} - a_{m-2} \cdots - a_0 = 0} \qquad (5.3)$$

Equ. (5.3) is equivalent to say that $\lambda_1 = 1$ is a root of the charact. polyn.

$$\mathcal{P}(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_0$$

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Let's consider the equation (discrete)

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \cdot\cdot + a_0 w_{i-(m-1)} \qquad (6.1)$$

independly of IVP (5.1).

and also consider the corresponding charact. polyn.

$$\mathcal{P}(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \dots - a_0$$

There, $w_n = \lambda^n$, for each '$n$' is a soln. if $\lambda$ is a root of $\mathcal{P}(\lambda)$

Since Subst. it into (6.1)

$$\lambda^{i+1} - \left(a_{m-1}\lambda^i + a_{m-2}\lambda^{i-1} + \cdot\cdot + a_0 \lambda^{i-(m-1)}\right) =$$

$$= \lambda^{i-(m-1)}\left[\lambda^m - a_{m-1}\lambda^{m-1} + \cdot\cdot + a_0\right] \overset{\text{if } \lambda \text{ is root of } \mathcal{P}.}{=} 0$$

Similar to what happens in ODE's, if $\lambda_1, \dots, \lambda_m$ are distinct roots of $\mathcal{P}(\lambda)$, then every solution of (6.1) can be expressed as

$$w_n = \sum_{i=1}^{m} c_i \lambda_i^n \qquad \text{and } c_i\text{'s are unique.}$$

In our particular case, $\lambda_1 = 1$ is a root in order for $w_n \equiv \alpha$, for all $n$

and

$$w_n = \alpha + \sum_{i=2}^{m} C_i \lambda_i^n$$

In general, $C_i$'s are not zero due to roundoff errors

as $n$ grows $\lambda_i^n$ will grow if $|\lambda_i| > 1$. Therefore,

for stable method, we need $|\lambda_i| \le 1$, $i = 2, .., m$.