Tridiagonal Algorithms. Crout Factorization

A 4x4 example

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}$$

Crout proposed a decomposition of A as $A=LU$, where

$$L=\begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{pmatrix}, \quad U=\begin{pmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then,

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} l_{11} & l_{11}u_{12} & 0 & 0 \\ l_{21} & l_{21}u_{12}+l_{22} & l_{22}u_{23} & 0 \\ 0 & l_{32} & l_{32}u_{23}+l_{33} & l_{33}u_{34} \\ 0 & 0 & l_{43} & l_{43}u_{34}+l_{44} \end{pmatrix}$$

Therefore,

a) $l_{11} = a_{11}$

b) $l_{i,i-1} = a_{i,i-1}, \quad i = 2,..4$

c) $l_{11} u_{12} = a_{12} \Rightarrow u_{12} = a_{12}/l_{11}$

d) $l_{21} u_{12} + l_{22} = a_{22} \Rightarrow l_{22} = a_{22} - l_{21} u_{12}$

e) $l_{22} u_{23} = a_{23} \Rightarrow u_{23} = a_{33}/l_{22}$

f) $l_{32} u_{23} + l_{33} = a_{33} \Rightarrow l_{33} = a_{33} - l_{32} u_{23}$

g) $l_{33} u_{34} = a_{34} \Rightarrow u_{34} = a_{34}/l_{33}$

h) $l_{43} u_{34} + l_{44} = a_{44} \Rightarrow l_{44} = a_{44} - l_{43} u_{34}$

$$\begin{pmatrix} l_{11} & l_{11} u_{12} & 0 & 0 \\ l_{21} & l_{21} u_{12} + \boxed{l_{22}} & l_{22} \boxed{u_{23}} & 0 \\ 0 & l_{32} & l_{32} u_{23} + \boxed{l_{33}} & l_{33} \boxed{u_{34}} \\ 0 & 0 & l_{43} & l_{43} u_{34} + \boxed{l_{44}} \end{pmatrix}$$

<u>Algorithm</u> $A_{n\times n}$ Finding of L and U.

$$\ell_{11} = a_{11}$$
$$u_{12} = a_{12}/\ell_{11}$$

For $i = 2, \dots, n-1$

$$\ell_{i,i-1} = a_{i,i-1}$$
$$\ell_{i,i} = a_{ii} - \ell_{i,i-1}\, u_{i-1,i}$$
$$u_{i,i+1} = a_{i,i+1}/\ell_{ii}$$

end

$$\ell_{n,n-1} = a_{n,n-1}$$
$$\ell_{n,n} = a_{nn} - \ell_{n,n-1}\, u_{n-1,n}$$

<u>System solution</u> $A\vec{x} = \vec{b} \iff LU\vec{x} = \vec{b}$

In two steps: (I) $L\vec{y} = \vec{b}$

(II) $U\vec{x} = \vec{y}$

(I)
$$\begin{pmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ & \ddots & \ddots & \\ & & \ell_{n,n-1} & \ell_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

<u>Forward Subst.</u>

$\ell_{11} y_1 = b_1 \Rightarrow y_1 = b_1/\ell_{11}$

$\ell_{21} y_1 + \ell_{22} y_2 = b_2 \Rightarrow$

$\Rightarrow y_2 = \dfrac{b_2 - \ell_{21} y_1}{\ell_{22}}$

$\vdots$

$y_i = \dfrac{b_i - \ell_{i,i-1} y_{i-1}}{\ell_{ii}}$ $\quad i = 2, \dots n$

(II) $U\vec{x} = \vec{y}$.

$$\begin{pmatrix} 1 & u_{12} & 0 & \cdots & 0 \\ 0 & 1 & u_{23} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 1 & u_{n-1,n} \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \\ \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}$$

<u>Backward Subst.</u>

$x_n = y_n$

$x_{n-1} + u_{n-1,n} x_n = y_{n-1} \Rightarrow x_{n-1} = y_{n-1} - u_{n-1,n} x_n$

$\vdots$

In general, $x_i = y_i - u_{i,i+1} x_{i+1}$

$i = n-1, .., 1$

<u>Advantage</u>:

$\mathcal{O}(n)$ operations

vs

$\mathcal{O}(n^3/3)$ operations in Gauss elimination.