Summarizing Example 1.

Unsteady heat conduction in a rod.

Assumptions: i) Rod is laterally perfectly insulated.

ii) Rod is uniformly heated. thermal energy density varies only from one cross section to another.

$v(x,t)$: Temperature at cross section located in x at time t.

$e(x,t)$: Thermal energy density.

$$e(x,t) = C(x)\ \rho(x)\, v(x,t)$$

$Q(x,t)$: Heat energy per unit of volume, generated or taken away inside the slice

$\phi(x,t)$: Heat flux. Flow of energy per unit of time and per unit of surface area.

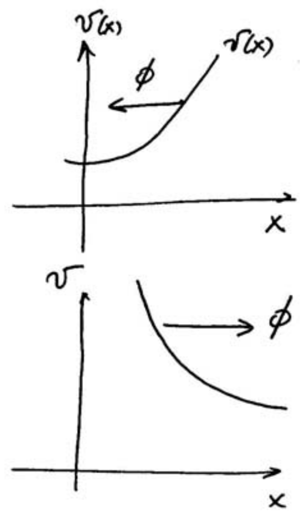
Principle of conservation of energy leads to

$$\frac{\partial}{\partial t}\left(e(\bar{x},t)\,A\,\Delta x\right) = \phi(x,t)A - \phi(x+\Delta x,t)A + Q(\bar{x},t)A\,\Delta x$$

Dividing by $A\Delta x$ and taking $\lim_{\Delta x \to 0}$, we arrive to

$$\boxed{\frac{\partial e}{\partial t}(x,t) = -\frac{\partial \phi}{\partial x}(x,t) + Q(x,t)} \qquad (1)$$

Fourier's law of heat conduction:



$$\phi(x,t) = -K_0 \frac{\partial v}{\partial x}$$

Substitution in (1) leads to

$$c\rho \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial v}{\partial x}\right) + Q(x,t)$$

If $K_0(x) \equiv$ constant

↳ Thermal conductivity

then

$$\boxed{\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2} + \frac{Q}{c\rho}} \quad (2), \qquad 0 < x < L,\ t > 0$$

It is known.

they are given.

If No sources or sinks : $Q = 0$.

$$\boxed{\frac{\partial v}{\partial t} = k \frac{\partial^2 v}{\partial x^2}} \qquad 0 < x < L,\ t > 0 \qquad (3)$$

k : Thermal diffusivity.

Fourier Laws:

$$C(x)\,\rho(x)\,\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + Q(x,t).$$

$C(x)$, $\rho(x)$, $Q(x,t)$: Known

(x,t), $\phi(x,t)$: Unknown. $\begin{cases} \phi > 0 \Rightarrow \text{Energy flows right} \\ \phi < 0 \Rightarrow \text{" " left} \end{cases}$

Fourier's law establishes a relationship between these two unknowns.

This Law should include familiar qualitative properties as

1) No flow if $V(x,t)$ constant.
(Temperature)

2) Heat energy flows from hotter regions to colder regions.

3) Greater temperature differences implies greater flow of heat energy.

4) Flow of heat energy depends on the material.

Therefore, a good law is given by
(or relationship)

$$\boxed{\phi(x,t) = -K_0(x)\,\frac{\partial u}{\partial x}(x,t)} \qquad (*)$$

Explain why the previous (4) Conditions are Satisfied.

<u>Exercise</u>: (add to the homework in section 1.2)
Find the physical units of $K_0(x)$. in terms of Energy (E), Length (L), Temperature (T), time (t).

<u>Diffusion of Chemical Pollutant:</u>

$u(x,t)$: density of chemical $[u] = \dfrac{\text{Amount of chemical}}{L^3}$

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} \qquad \Rightarrow \qquad \boxed{\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}}$$ Same as Heat conduction.

$$\begin{cases} \phi(x,t) = -k\,\dfrac{\partial u}{\partial x} \\ \text{Fick's Law of diffusion} \end{cases}$$

(3) is called equation of heat conduction.
It is of "parabolic type". It also models a diffusion process.

In order to solve (2) or (3), we need to impose two boundary conditions (BC's) and one initial condition (I.C.).

For example,

$$\begin{cases} v(0,t) = A(t) \\ v(L,t) = B(t) \end{cases} \quad \text{Dirichlet conditions.}$$

or

$$\begin{cases} \dfrac{\partial v}{\partial x}(0,t) = C(t) \\ \dfrac{\partial v}{\partial x}(L,t) = D(t) \end{cases} \quad \text{Neumann Condition}$$

or

$$\begin{cases} \left[\alpha_1 \dfrac{\partial v}{\partial x} + \beta_1 v\right](0,t) = H(t) \\ \left[\alpha_2 \dfrac{\partial v}{\partial x} + \beta_2 v\right](L,t) = G(t) \end{cases} \quad \text{Robin conditions}$$

or any combination of them. Discuss physical meaning!

1.3 Boundary Conditions.

A) **Prescribed Temperature:**

$$V(a,t) = A(t).$$

B) **Insulated boundary:**

$$\phi(a,t) = -K_0(a)\frac{\partial V}{\partial x}(a,t) = f(t).$$

or simply $\boxed{\frac{\partial V}{\partial x}(a,t) = g(t)}$

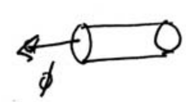
C)

Newton's Law of Cooling:

$$\phi(0,t) = -K_0(0)\frac{\partial V}{\partial x}(0,t) = -H\left[V(0,t) - \underbrace{V_B(t)}_{\text{Surrounding rod temperature}}\right].$$

(I) If rod temperature > Surrounding at $x=0$

$\Rightarrow \quad v(0,t) - v_B(t) > 0 \Rightarrow \phi(0,t) < 0$ heat flows out of rod



(II) If rod temp < Surrounding at $x=0$ $\Rightarrow v(0,t) - v_B(t) < 0 \Rightarrow \phi(0,t) > 0$

Heat flows from the surrounding to the rod.

Plus an initial Condition:

$$v(x,0) = f(x), \quad 0 < x < L.$$

The set of equations:

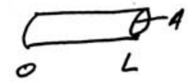
$$\begin{cases} \dfrac{\partial v}{\partial t} = k \dfrac{\partial^2 v}{\partial x^2} + \dfrac{Q}{c\rho}, & 0 < x < L, \ t > 0 \\ v(0,t) = A, & t > 0 \\ \dfrac{\partial v}{\partial x}(L,t) = 0, & t > 0 \\ v(x,0) = f(x), & 0 < x < L \end{cases}$$

form an initial boundary value problem (IBVP).

there are several analytic techniques to obtain its solution

Derivation of heat equation: Integral approach

Constant thermal properties: c, ρ, K_0



Constant cross sectional area: A.

$u(x,t)$: Temperature.

$$\frac{d}{dt}\int_0^L c\rho\, u(x,t)\cancel{A}\,dx = \phi(0,t)\cancel{A} - \phi(L,t)\cancel{A} + \int_0^L Q(x,t)\cancel{A}\,dx$$

Using Fund. Theorem of Calculus

$$c\rho\int_0^L \frac{\partial u}{\partial t}\,dx = -\int_0^L \frac{\partial \phi}{\partial x}(x,t)\,dx + \int_0^L Q(x,t)\,dx$$

$$\Rightarrow \int_0^L \left[c\rho\frac{\partial u}{\partial t}(x,t) + \frac{\partial \phi}{\partial x}(x,t) + Q(x,t)\right]dx = 0$$

Since interval $[0,L]$ is arbitrary and $\phi(x,t) = -K_0\frac{\partial u}{\partial x}$,

then $\boxed{c\rho\frac{\partial u}{\partial t} = K_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)}$ ✓

Important differential operators:

1) Divergence.

If $\vec{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$

$\vec{x} = (x,y,z)$

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x}(\vec{x}) + \frac{\partial F_2}{\partial y}(\vec{x}) + \frac{\partial F_3}{\partial z}(\vec{x}).$$

$$\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longrightarrow \vec{F}(x,y,z).$$

2) Gradient

If $f(x,y,z)$ is a scalar function of 3 variables

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longrightarrow f(x,y,z)$$

$$\nabla f(\vec{x}) = \left(\frac{\partial f}{\partial x}(\vec{x}), \frac{\partial f}{\partial y}(\vec{x}), \frac{\partial f}{\partial z}(\vec{x}) \right)$$

3) Laplacian.

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Proof.-

$$\nabla \cdot (\nabla f) = \nabla \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) =$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \checkmark$$

Divergence Theorem.

Concept of Region:- An open set containing all or some of the points forming its boundary.

Def.- A closed surface $\partial\Omega$ that consists of a finite number of smooth pieces joined together at the boundaries (curves defining boundaries) is called piecewise-smooth surface.

Def.- By a smooth surface S, we mean

$$S: \ \vec{r} = \vec{r}(u,v), \quad (u,v) \in D.$$

Such that $\vec{r}(u,v)$ is continuously differentiable and its unit normal vector $\hat{n}(\vec{x}_s)$ is continuous on S.

Theorem.- a) Ω is a bounded region.

b) $\partial\Omega$ is the closed piecewise smooth surface of Ω.

c) $F(\vec{x})$ is continuous on $\Omega \cup \partial\Omega$.

d) $F(\vec{x})$ is continuosly differentiable in Ω.

e) $\hat{n}(\vec{x}_s)$ is the unit outer normal vector to Ω at $\vec{x}_s$.

Then,

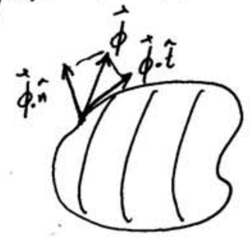
$$\iiint_{\Omega} \nabla \cdot \vec{F}(\vec{x})\, dv = \iint_{\partial\Omega} (\vec{F} \cdot \hat{n})(\vec{x}_s)\, ds.$$

In 2-D or 3-D, Consider an arbitrary region R

Want to study conservation of heat energy inside Region R bounded by surface S.

$$\text{Heat energy inside } R = \iiint_R c(x,y,z)\,\rho(x,y,z)\,v(x,y,z,t)\,dv$$

ϕ or heat flux is now a vector $\vec{\phi}(x,y,z,t)$.



$$\text{Flux of heat energy out of region } R = \iint_S \vec{\phi}(x,y,z,t)\cdot\hat{n}(x,y,z)\,ds$$

$$\overset{\text{Div thm}}{=} \iiint_R \nabla\cdot\phi(x,y,z,t)\,dv$$

Heat energy per unit of Vol. inside R = $Q(x,y,z,t)$.

Therefore, Conservation of energy inside R leads to

Ask ?

$$\frac{\partial}{\partial t}\iiint_R c\rho v\,dv = -\iint_S \vec{\phi}\cdot\hat{n}\,ds + \iiint_R Q(\vec{x},t)\,dv$$

$$= -\iiint_R \nabla\cdot\vec{\phi}\,dv + \iiint_R Q\,dv$$

Example. – Consider $f(x,y) = xe^{y}$

a) Find the directional derivative of $f(x,y)$ at $(2,0)$ in the direction $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $(1,0)$, $(0,1)$, $(1,2)$

b) In what direction does f have the maximum rate of change?. What is the maximum rate of change?

$$D_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)} f = \nabla f^{(2,0)} \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(e^{y}, xe^{y}\right)\Big|_{(2,0)} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$= (1,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

$$D_{\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)} f = \nabla f(2,0) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = (1,2) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}.$$

$$D_{(1,0)} f = \frac{\partial f}{\partial x}(2,0) = e^{0} = 1$$

$$D_{(0,1)} f(2,0) = \frac{\partial f}{\partial y}(2,0) = 2.$$

Largest value : $\boxed{D_{\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)} f(2,0) = \sqrt{5}}$

b) In general, for $\vec{\omega}$ s.t. $\|\vec{\omega}\|=1$

$$D_{\vec{\omega}} f(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{\omega}$$

but

$$\nabla f(x_0,y_0) \cdot \vec{\omega} = \|\nabla f(x_0,y_0)\| \overbrace{\|\vec{\omega}\|}^{=1} \cos\theta$$

$$= \|\nabla f(x_0,y_0)\| \cos\theta$$

<u>Largest value when $\theta = 0$.</u>

$\Rightarrow$ $\vec{\omega}$ is in the direction of $\nabla f(x_0,y_0)$.

In our previous example,

$$\nabla f(2,0) = \left(e^y, xe^y\right)\Big|_{(2,0)} = (1,2).$$

So $(1,2)$ is the direction of maximum increase. for $f(x,y)$ at the point $(2,0)$

Maximum rate of change:

$$D_{\frac{\nabla f(2,0)}{\|\nabla f\|}} f(2,0) = D_{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)} f(2,0) = \nabla f(2,0) \cdot \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

$$= \sqrt{5}.$$

Fourier's Law of heat Conduction in Higher dimensions

In 1-d

$$\text{Flux} = \phi(x,t) = -K_0(x)\,\frac{\partial u}{\partial x}(x,t).$$

In 2-d or 3-d.

The flux is a Vector $\vec{\phi}(\vec{x},t) = \vec{\phi}(x,y,z,t)$.

The Question is in what direction the heat energy is naturally flowing.

<u>answer</u>: For isotropic materials, "heat flows from hot to cold in the direction in which temperature differences are greatest."

So

$$\boxed{\vec{\phi}(\vec{x},t) = -K_0(\vec{x})\,\nabla u}$$

or

$$\iiint_R \left[c\rho \frac{\partial v}{\partial t} + \nabla\cdot\phi - Q\right] dv = 0$$

True for any region R, then

$$c\rho \frac{\partial v}{\partial t} = -\nabla\cdot\phi + Q$$

Again, Fourier's Law of Heat conduction:

$$\phi = -K_0 \nabla v$$

$$\Rightarrow \quad \boxed{c\rho \frac{\partial v}{\partial t} = \nabla\cdot(-K_0 \nabla v) + Q.} \qquad (2.1)$$

If thermal conductivity K_0 is constant and $Q=0$. (no sources of Heat energy)

$$\frac{\partial v}{\partial t} = k\ \nabla\cdot(\nabla v)$$

then $\boxed{\frac{\partial v}{\partial t} = k\nabla^2 v}$ Heat or diffusion equ. (2.2)

<u>Possible BC's.</u>: $v(x,y,z,t) = T(x,y,z,t)$ on boundary S.

$$\frac{\partial v}{\partial n} = \nabla v\cdot\hat{n}\,(x,y,z,t) = H(x,y,z,t) = \begin{cases} \neq 0 \\ = 0, \text{ insulated.}\end{cases}$$

<u>IC's</u>: $v(x,y,z,0) = f(x,y,z)$. <u>Equations of elliptic type.</u>

Steady state + No heat sources + Constant thermal properties in (2.1) leads to

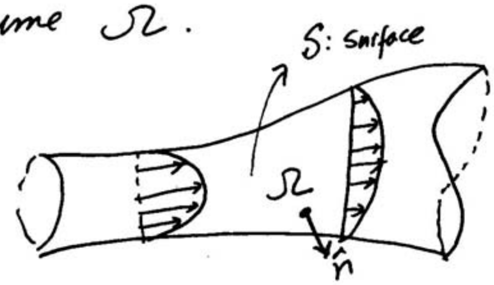
$$\nabla^2 v = \frac{-Q}{K_0} \quad \text{Poisson's equ.}$$

If $Q=0$ $\quad \nabla^2 v = 0,$ Laplace's equation

Discuss analytic techniques and limitations.!

Example 2.– Continuity equation

Conservation of mass in an arbitrarily shaped closed volume Ω.



$\vec{v}(\vec{x},t)$: Velocity , $\hat{n}(\vec{x})$: outward normal (unit vector).

$\rho(\vec{x},t)$: density of mass.

Principle of Conservation of mass:

Rate of change of mass inside Ω per unit of time. = Mass inflow per unit of time through surface S.

$$\frac{\partial}{\partial t}\iiint_{\Omega}\rho(\vec{x},t)\,dv = -\iint_{S}\rho\,\vec{v}\cdot\hat{n}\,ds \qquad (1)$$

Applying Gauss theorem:

$$\iint_S \left[\rho(\vec{x},t)\, \vec{v}(\vec{x},t)\right] . \hat{n}(\vec{x})\, ds = -\iiint_\Omega \nabla . \left(\rho(\vec{x},t)\, \vec{v}(x,t)\right) dv$$

Therefore, (1) transforms into

$$\iiint_\Omega \left[\frac{\partial \rho}{\partial t}(\vec{x},t) + \nabla . (\rho \vec{v})(\vec{x},t)\right] dv = 0$$

Since the domain Ω is arbitrary

$$\boxed{\frac{\partial \rho}{\partial t}(\vec{x},t) + \nabla . (\rho \vec{v})(\vec{x},t) = 0} \qquad (2)$$

Equation of continuity.