# Maximum Principle.

Consider the BVP for Laplace equation over a circular region centered at point $\vec{p}$ in $\mathbb{R}^2$.

$$\begin{cases} \nabla^2_{r,\theta} v = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial v}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0, & (0.1) \\ \qquad \vec{x} \in \Omega = \{\vec{x} : \|\vec{x}-\vec{p}\| \le r_0\} & \\ v(r_0,\theta) = f(\theta), \quad 0 \le \theta < 2\pi. & (0.2) \end{cases}$$

The solution for this BVP can be obtained by separation of variables. It is given by

$$\boxed{v(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)} \quad (1.1)$$

where

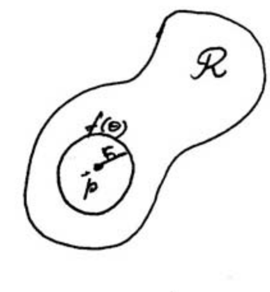
$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta$$


$$A_n r_0^n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\, d\theta, \qquad B_n a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\, d\theta$$

Thus, $\boxed{v(0,\theta) = A_0}$

Now, let $\mathcal{R}$ be an arbitrary region where

$$\nabla^2 v = 0, \qquad \vec{x} \in \mathcal{R} \subset \mathbb{R}^2$$




Consider an arbitrary point $\vec{p} \in \mathcal{R}$, and let $B_r(\vec{p})$ be a circle of radius $r_0$ centered at $\vec{p}$, which is contained in $\mathcal{R}$.

When $r_0 = 0$, we are at point $\vec{p}$

therefore, from (1.1)

$$\underset{\overset{\|}{v(0,\theta)}}{v(\vec{p})} = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta \overset{\text{peridiocity}}{=\!=} \frac{1}{2\pi}\int_0^{2\pi} v(r_0,\theta)\, d\theta$$

Thus, the solution of $v$ at $\vec{x} = \vec{p}$ (arbitrary), $v(\vec{p})$, is the average of the values at any circle, with center at $\vec{p}$, lying in $\mathcal{R}$. This is the <u>Mean Value Theorem</u>.

Obviously, if $f(\theta) \equiv \text{const.} = K$

$v(r,\theta) \equiv K$ is a solution of BVP (0.1), (0.2).

## Maximum Principle Theorem.-

The solution of Laplace equation on a closed region $\mathcal{R}$ with boundary $\partial\mathcal{R}$, reach its maximum and minimum at the boundary $\partial\mathcal{R}$.

Proof.-

Assume there is $\vec{p}^*$ in the interior of $\mathcal{R}$, such that $v(\vec{p}^*) > v(\vec{p})$ (*), for all $\vec{p}$ in $\mathcal{R}$.

Construct a circle of radius $r_0$, $B_{r_0}(\vec{p}^*) \subset \mathcal{R}$ centered at $\vec{p}^*$. Then,

$$v(\vec{p}^*) = v(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} v(r_0,\theta)\,d\theta$$

and it should be $\vec{p}_0$ in $\partial B_{r_0} \subset \mathcal{R}$ such that $v(\vec{p}_0) \geq v(\vec{p})$ contradicting (*).

Therefore, there is $\vec{p}_0$ in $\partial\mathcal{R}$ such that

$$v(\vec{p}_0) \geq v(\vec{p}), \text{ for all } \vec{p} \text{ in } \mathcal{R}.$$

Similarly, we can prove that there is $\vec{\hat{p}}$ in $\partial\mathcal{R}$ such that $v(\vec{\hat{p}}) \leq v(\vec{p})$, for all $\vec{p}$ in $\mathcal{R}$.

Corollary.- If the BVP for the Poisson equation

$$\begin{cases} \nabla^2 v = f(\vec{x}), & \vec{x} \text{ in the interior of } \mathcal{R} \\ v(\vec{x}) = g(\vec{x}), & \vec{x} \text{ in } \partial\mathcal{R}. \end{cases}$$

has a solution, then this solution is unique.

Let $v(\vec{x})$ and $u(\vec{x})$ be two diff. solutions and define $w = v - u$.

then $\nabla^2 w = \nabla^2 v - \nabla^2 u = f(\vec{x}) - f(\vec{x}) = 0$, $\vec{x}$ interior $\mathcal{R}$

also $w(\vec{x}) = v(\vec{x}) - u(\vec{x}) = g(\vec{x}) - g(\vec{x}) = 0$, $\vec{x}$ in $\partial\mathcal{R}$.

Since max and min. are reached at the boundary $\partial\mathcal{R}$

$$w(\vec{x}) \equiv 0 \Rightarrow v(\vec{x}) = u(\vec{x}).$$

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DISCRETE MAX. PPLE.-

Consider again Laplace's equation

$$\nabla^2 u = 0, \quad \vec{x} \text{ in interior of } \mathcal{R}$$

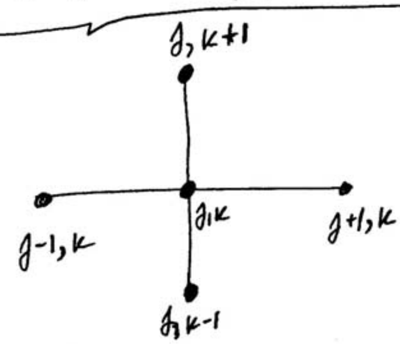
and the centered finite diffce approx.

$$\boxed{-v_{jk} + \theta_x \left(v_{j+1,k} + v_{j-1,k}\right) + \theta_y \left(v_{j,k+1} + v_{j,k-1}\right) = 0} \quad (4.1)$$

If $\Delta x = \Delta y \Rightarrow \theta_x = \dfrac{\Delta x^2}{\Delta x^2 + \Delta y^2} = \dfrac{1}{2} = \theta_y$.

Therefore, (4.1) reduces to

$$\boxed{U_{jk} = \frac{1}{4}\left[ U_{j+1,k} + U_{j-1,k} + U_{j,k+1} + U_{j,k-1} \right]} \quad (5.1)$$




Value at $(x_j, y_k)$, $U_{jk}$ is the average of the neighbor points $(x_{j-1}, y_k)$, $(x_{j+1}, y_k)$, $(x_j, y_{k+1})$, and $(x_j, y_{k-1})$.

Therefore, the max and min. approx. obtained using (5.1) is reached at discrete points in the boundary. In fact,

$$U_{jk} \le 4 * \frac{1}{4} \max \{ U_{j+1,k}, U_{j-1,k}, U_{j,k+1}, U_{j,k-1} \}$$
$$= \max \{ U_{j+1,k}, \ldots, U_{j,k-1} \}$$

for all $(x_j, y_k)$ in $R$.

This inequality also holds in the more general case

$$-v_{jk} + \theta_x\left(v_{j+1,k} + v_{j-1,k}\right) + \theta_y\left(v_{j,k+1} + v_{j,k-1}\right) = 0.$$

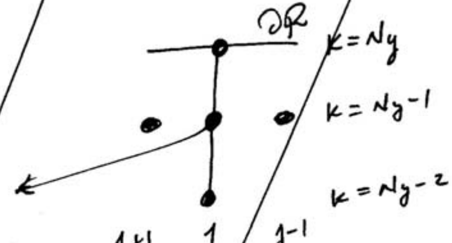
In fact,

$$v_{jk} = \theta_x(\quad) + \theta_y(\quad) \le$$

$$\le \max\{v_{j+1,k}, \dots, v_{j,k-1}\}\underbrace{(2\theta_x + 2\theta_y)}_{=1} =$$

$$= \max\{\quad\}, \text{ for all } (x_j, y_k) \text{ in } R$$

Now, the strict inequality is satisfied if for example a homogeneous Dirichlet condition is imposed in any part of the boundary $\partial R$. In fact




$(x_j$

$$\therefore\quad v_{j,N_y-1} = \theta_x\left(v_{j+1,N_y-1} + v_{j-1,N_y-1}\right) + \theta_y \overset{0}{v_{j,N_y}} + \theta_y v_{j,N_y-2}$$

$$\Rightarrow\quad v_{j,N_y-1} \le \max\left(v_{j+1,N_y-1}, v_{j-1,N_y-1}, v_{j,N_y-2}\right)\underbrace{(2\theta_x + \theta_y)}_{<1}$$

$$< \max(\quad)$$

Finite diffce (4.1) can be written as

$$C_0 v_{jk} - \left( C_{1,0} v_{j+1,k} + C_{-1,0} v_{j-1,k} + C_{0,1} v_{j,k+1} + C_{0,-1} v_{j,k-1} \right)$$

where $C_0 = 1$, $C_{1,0} = C_{-1,0} = \theta_x$ and $C_{0,1} = C_{0,-1} = \theta_y$

In general, we can define a finite diffce operator

$$L_\Delta v_{jk} = C_0 v_{j,k} - \sum_{|\vec{s}| \le S,\ |\vec{s}| \ne 0} C_{\vec{s}}\, v_{\vec{Q}_s}$$

to approximate a linear elliptic problem

$$\mathcal{L} u = 0,$$

where $\vec{S} = (s_1, s_2)^T$, the condition $|\vec{s}| \ne 0 \Rightarrow s_1 \ne 0$ or $s_2 \ne 0$

$s_1, s_2$ integers

and $\vec{Q}_s = (j+s_1, k+s_2)$.

This definition allows to generalize the idea of the discrete max. Pple to more general finite difference scheme for elliptic problems. (See Flaherty notes).

## Special matrices.

Consider the numerical scheme "backward Euler method" used to approximate the heat conduction equation BT-CS about the point $(x_j, t_{n+1})$.

$$\frac{U_j^{n+1} - U_j^{n}}{\Delta t} = \sigma \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{\Delta x^2}$$

or

$$-rU_{j-1}^{n+1} + (1+2r)U_j^{n+1} - rU_{j+1}^{n+1} = U_j^{n}, \qquad j = 1, 2, .., J-1$$

Assuming BC's. $U_0^{n+1} = f^{n+1}$, $U_J^{n+1} = g^{n+1}$

We obtain the following linear system: $\boxed{A\vec{U}^{n+1} = \vec{U}^{n} + r\vec{f}^{n}}$

Implicit scheme.

where

$$A = \begin{pmatrix} 1+2r & -r & 0 & 0 & 0 & 0 & \cdots & 0 \\ -r & 1+2r & -r & 0 & 0 & 0 & \cdots & 0 \\ 0 & -r & 1+2r & -r & 0 & \cdots & \cdots & 0 \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & -r & 1+2r & -r \\ & & & & & & -r & 1+2r \end{pmatrix}$$

The matrix $A$ is banded, tridiagonal matrix, but also it has other properties. For example,

$$|a_{ii}| > |a_{i-1,i}| + |a_{i,i+1}|$$

Def.– $A_{n\times n}$ is said strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad \text{for all } i=1,2,..,n$$

Examples: a) Previous matrix $A$ of Euler implicit scheme for heat conduction.

b) The matrix corresponding to 5 points discretiz for Laplace's equation (Cent$_x$-Cent$_y$) is diagonally dominant but not strict.

Thm.– $A_{n\times n}$ strictly diagonally dominant $\Rightarrow$ $A$ is non-singular.

Proof.– If $A$ were singular $\Rightarrow$ there exists $\vec{x}\neq\vec{0}$ such that $A\vec{x}=\vec{0}$.

If $\vec{x}=(x_1,x_2,..,\overset{\neq 0}{x_k},.,x_n)$ and

$$0 < |x_k| = \max_{j=1,..,n} |x_j|$$

Therefore,

$$\sum_{j=1}^{n} a_{ij} x_j = 0, \quad \text{for all } i = 1,..,n$$

$$\overset{i=k}{\Longrightarrow} a_{kk} x_k = - \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} x_j$$

$$\Rightarrow \quad |a_{kk}|\,|x_k| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|\,|x_j|$$

$$\Rightarrow \quad |a_{kk}| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| \overbrace{\frac{|x_j|}{|x_k|}}^{\leq 1} \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|$$

Contradicting the strictly diagonally dominant hypothesis.

Therefore, $A$ is non-singular.

**Def.**- $A_{n \times n}$ is positive definite if

i) $A$ is Symmetric

ii) For all $\vec{x} \neq \vec{0}$, $\vec{x}^T A \vec{x} > 0$.

**Thm.**- $A_{n \times n}$ is positive definite iff all eigenvalues of $A$ are positive.

**Proof.**- $(\rightarrow)$ $A$ positive definite $\Rightarrow$ $0 < \vec{x}^T A \vec{x} =$

$= \vec{x}^T \lambda \vec{x}$, if $\vec{x}$ eigenvector with eigenvalue $\lambda$.

Thus

$$0 < \vec{x}^T A \vec{x} = \lambda \|\vec{x}\|^2 \Rightarrow \lambda > 0$$

Thm.— A positive definite $\Rightarrow$ A is non-singular.

Proof.—

If A were singular $\Rightarrow$ there exists $\vec{x} \neq 0$ such that $A\vec{x} = \vec{0} \Rightarrow \vec{x}^T A \vec{x} = 0 \Rightarrow A$ is not positive definite.

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The matrix obtained for our Dirichlet BVP for Laplace's equation is positive definite. This can be shown by proving that all eigenvalues are positive (not easy).

Therefore, the discrete linear system has a unique solution.