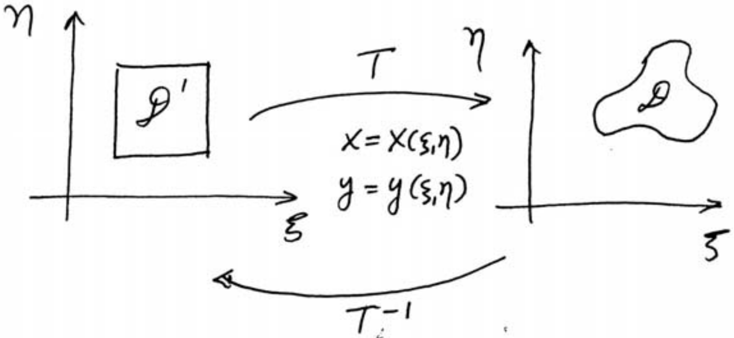
Generalized Curvilinear Coordinates.

Consider the transformation T from the computational domain $\mathcal{D}'$ into the physical domain $\mathcal{D}$.



$$x = x(\xi,\eta), \quad y = y(\xi,\eta) \qquad (1)$$

$$\xi = \xi(x,y)$$
$$\eta = \eta(x,y) \qquad (2)$$

Assuming that (Inverse transformation thm.)

$$J(\xi,\eta) = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} \neq 0, \quad \text{for all } (\xi,\eta) \in \mathcal{D}'$$

then, the inverse transformation (2) exists for all $(\xi,\eta) \in \mathcal{D}$

Thm.- If $J(\xi,\eta) \neq 0$, for all $(\xi,\eta) \in \mathcal{D}'$

then, the Jacobian matrix of the inverse transformation

$$\hat{J} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$$

Satisfies

$$\hat{J}^{-1} = J \equiv \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}$$

Proof.– From eqns. (1)-(2) defining the transformation T, T^{-1} we obtain the equations:

$$F(x,y) = x - x\left(\xi(x,y), \eta(x,y)\right) = 0 \qquad (2.1)$$

$$G(x,y) = y - y\left(\xi(x,y), \eta(x,y)\right) = 0, \quad \text{for all } (x,y) \in \mathcal{D}.$$

Then,

$$F_x(x,y) = 1 - x_\xi(\xi,\eta)\,\xi_x - x_\eta(\xi,\eta)\,\eta_x = 0 \qquad (2.2)$$

$$G_x(x,y) = 0 - y_\xi(\xi,\eta)\,\xi_x - y_\eta(\xi,\eta)\,\eta_x = 0, \quad (x,y) \in \mathcal{D}$$

In matrix form:

$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} \xi_x \\ \eta_x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (3)$$

Analogously,

$$F_y(x,y) = 0 - x_\xi\,\xi_y - x_\eta\,\eta_y = 0$$

$$G_y(x,y) = 1 - y_\xi\,\xi_y - y_\eta\,\eta_y = 0$$

In matrix form:

$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{bmatrix} \xi_y \\ \eta_y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (4)$$

Combining (3) and (4)

$$\begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$\hat{J} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = J^{-1} = \frac{1}{J}\begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}$$

Thus,

$$\boxed{\begin{aligned} \xi_x &= \frac{y_\eta}{J}, & \xi_y &= -\frac{x_\eta}{J} \\ \eta_x &= \frac{-y_\xi}{J}, & \eta_y &= \frac{x_\xi}{J} \end{aligned}} \qquad (5).$$

Second order derivatives: ξ_{xx}, η_{xx}, ξ_{yy}, and η_{yy} in terms of $x_\xi, x_\eta, y_\xi, y_\eta, x_{\xi\xi}, y_{\xi\xi}, \ldots$, etc.

Taking another derivative with respect to x from (2.2), we obtain

$$F_{xx} = -\left(x_{\xi\xi}\,\xi_x^2 + \underline{x_{\xi\eta}}\,\xi_x\eta_x + x_\xi\,\xi_{xx} + \underline{x_{\eta\xi}}\,\eta_x\xi_x + x_{\eta\eta}\eta_x^2 + x_\eta\eta_{xx}\right) = 0$$

$$G_{xx} = -\left(y_{\xi\xi}\,\xi_x^2 + 2\,y_{\xi\eta}\,\xi_x\eta_x + y_\xi\,\xi_{xx} + y_{\eta\eta}\,\eta_x^2 + y_\eta\,\eta_{xx}\right) = 0$$

Then, the following system of equations for ξ_{xx} and η_{xx} are obtained

$$x_\xi \xi_{xx} + x_\eta \eta_{xx} = -\left(x_{\xi\xi}\xi_x^2 + 2x_{\xi\eta}\xi_x\eta_x + x_{\eta\eta}\eta_x^2\right)$$

$$= -\frac{1}{J^2}\left(x_{\xi\xi}y_\eta^2 - 2x_{\xi\eta}y_\xi y_\eta + x_{\eta\eta}y_\xi^2\right)$$

$$y_\xi \xi_{xx} + y_\eta \eta_{xx} = -\frac{1}{J^2}\left(y_{\xi\xi}y_\eta^2 - 2y_{\xi\eta}y_\xi y_\eta + y_{\eta\eta}y_\xi^2\right)$$

Solving this nonhomogeneous system of equations, we obtain

$$\xi_{xx} = \frac{1}{J^3}\begin{vmatrix} A & x_\eta \\ B & y_\eta \end{vmatrix} = \frac{A y_\eta - B x_\eta}{J^3} \qquad (6)$$

where

$$A \equiv x_{\xi\xi}y_\eta^2 - 2x_{\xi\eta}y_\xi y_\eta + x_{\eta\eta}y_\xi^2 . \qquad (6.1)$$

$$B \equiv y_{\xi\xi}\xi_\eta^2 - 2y_{\xi\eta}y_\xi y_\eta + y_{\eta\eta}y_\xi^2 . \qquad (6.2)$$

and

$$\eta_{xx} = \frac{1}{J^3}\begin{vmatrix} x_\xi & A \\ y_\xi & B \end{vmatrix} = \frac{B x_\xi - A y_\xi}{J^3} \qquad (7)$$

Follow a similar procedure and show that

$$\xi_{yy} = \frac{C y_\eta - D x_\eta}{J^3}, \qquad \eta_{yy} = \frac{D x_\xi - C y_\eta}{J^3} \qquad (8)$$

where

$$C \equiv -\left(x_{\xi\xi}x_\eta^2 - 2x_{\xi\eta}x_\xi x_\eta + x_{\eta\eta}x_\xi^2\right)$$

$$D \equiv -\left(y_{\xi\xi}x_\eta^2 - 2y_{\xi\eta}x_\xi x_\eta + y_{\eta\eta}x_\xi^2\right) \qquad (9)$$

Thm.— If $J(\xi,\eta) \neq 0$, for all $(\xi,\eta) \in \mathcal{D}'$

Show that the inverse of the Laplace system of equations

$$\begin{cases} \xi_{xx} + \xi_{yy} = 0, \\ \eta_{xx} + \eta_{yy} = 0, \end{cases} \quad (x,y) \in \mathcal{D}.$$

Can be written as the Winslow's quasilinear elliptic system

$$\begin{cases} \alpha\, x_{\xi\xi} - 2\beta\, x_{\xi\eta} + \gamma\, x_{\eta\eta} = 0, \\ \alpha\, y_{\xi\xi} - 2\beta\, y_{\xi\eta} + \gamma\, y_{\eta\eta} = 0, \end{cases} \quad (\xi,\eta) \in \mathcal{D}'$$

Where $\alpha \equiv x_\eta^2 + y_\eta^2$, $\beta \equiv x_\eta x_\xi + y_\xi y_\eta$, $\gamma \equiv x_\xi^2 + y_\xi^2$

Laplace equation in terms of generalized coordinates.

First derivatives: $\hat{f}(\xi,\eta) = f(x(\xi,\eta), y(\xi,\eta))$

$$f(x,y) = \hat{f}(\xi(x,y), \eta(x,y))$$

$$\Rightarrow \quad f_x = \hat{f}_\xi\, \xi_x + \hat{f}_\eta\, \eta_x = \frac{1}{J}\left[\hat{f}_\xi\, y_\eta - \hat{f}_\eta\, y_\xi\right]$$

$$f_y = \hat{f}_\xi\, \xi_y + \hat{f}_\eta\, \eta_y = \frac{1}{J}\left[-\hat{f}_\xi\, x_\eta + \hat{f}_\eta\, x_\xi\right]$$

Thm.- If $T : \mathcal{D}' \to \mathcal{D}$ is invertible $\left(J(\xi,\eta) \neq 0 \;\; (\xi,\eta) \in \mathcal{D}' \right)$ then,

$$f_{xx}(x,y) = \frac{1}{J^2}\left[\hat{f}_{\xi\xi}\, y_\eta^2 - 2\,\hat{f}_{\xi\eta}\, y_\eta y_\xi + \hat{f}_{\eta\eta}\, y_\xi^2 \right]$$

$$+ \frac{A}{J^3}\left(y_\eta \hat{f}_\xi - y_\xi \hat{f}_\eta \right) + \frac{B}{J^3}\left(x_\eta \hat{f}_\eta - x_\eta \hat{f}_\xi \right)$$

$$f_{yy}(x,y) = \frac{1}{J^2}\left[\hat{f}_{\xi\xi}\, x_\eta^2 - 2\,\hat{f}_{\xi\eta}\, x_\eta x_\xi + \hat{f}_{\eta\eta}\, x_\xi^2 \right]$$

$$+ \frac{C}{J^3}\left(y_\eta f_\xi - y_\xi f_\eta \right) + \frac{D}{J^3}\left(x_\xi f_\eta - x_\eta f_\xi \right)$$

Corollary.- If $J(\xi,\eta) \neq 0$, $(\xi,\eta) \in \mathcal{D}'$, then

$$\nabla^2_{x,y} f = f_{xx} + f_{yy} =$$

$$= \frac{1}{J^2}\left[\alpha\,\hat{f}_{\xi\xi} - 2\beta\,\hat{f}_{\xi\eta} + \gamma\,\hat{f}_{\eta\eta} \right]$$

$$+ \frac{1}{J^3}\left(\alpha\, x_{\xi\xi} - 2\beta\, x_{\xi\eta} + \gamma\, x_{\eta\eta} \right)\left(y_\xi \hat{f}_\eta - y_\eta \hat{f}_\xi \right)$$

$$+ \frac{1}{J^3}\left(\alpha\, y_{\xi\xi} - 2\beta\, y_{\xi\eta} + \gamma\, y_{\eta\eta} \right)\left(x_\eta \hat{f}_\xi - x_\xi \hat{f}_\eta \right)$$

Corollary.- If $J(\xi,\eta) \neq 0$, $(\xi,\eta) \in \mathcal{D}'$ and $\begin{cases} \xi_{xx} + \xi_{yy} = 0 \\ \eta_{xx} + \eta_{yy} = 0 \end{cases}$

then,

$$\nabla^2_{x,y} f = \frac{1}{J^2}\left(\alpha\,\hat{f}_{\xi\xi} - 2\beta\,\hat{f}_{\xi\eta} + \gamma\,\hat{f}_{\eta\eta} \right).$$